Product (Re)forms
Part 1

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Based on work by…

Adan, Ahn, Akgun, Anton, Atar, Ayesta, Bodas, Bonald,

Busic, Caldentey, Comte, Doroudi, Dorsman, Gardner,

Gel, Gupta, Harchol-Balter, Hellemans, Hopp, Hurkens,

Hyytiä, Jonckheere, Kaplan, Kesslassy, Kleiner, Krzesinski,

Mairesse, Mathieu, Mendelson, Moyal, Perry, Righter,

Scheller-Wolf, van Houdt, Van Oyen, Velednitsky,

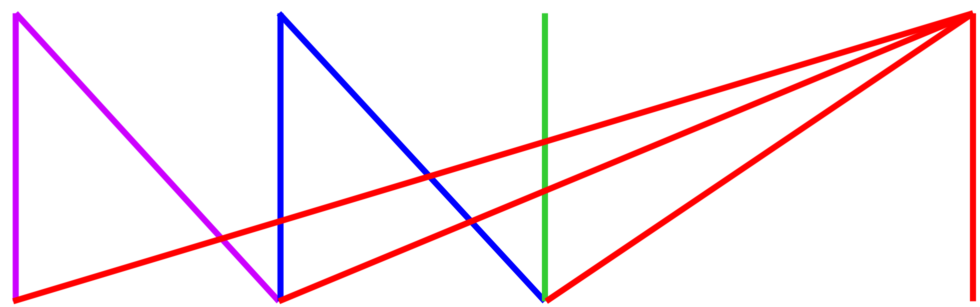
Verloop, Visschers, Weiss, Wolff, Zbarsky…

System Structure

Poisson arrivals: $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_J = \lambda$

Job classes: $1 \quad 2 \quad 3 \quad \ldots \quad J$

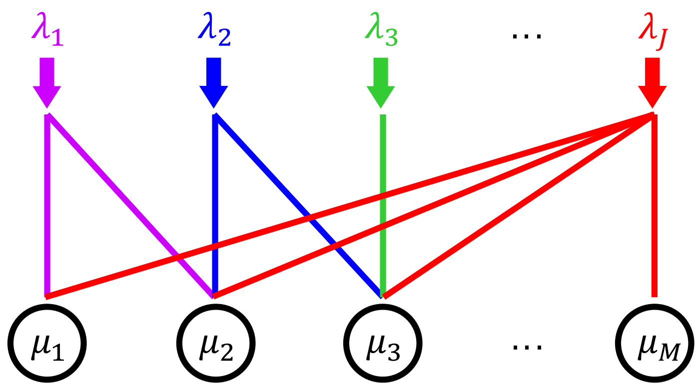
Bipartite matching structure:



Servers: $1 \quad 2 \quad 3 \quad \ldots \quad M$

Exponential service times: $\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_M = \mu$

System Structure



For stability, assume:

$$\lambda(A) < \mu(A)$$

for all job class subsets A

Notation

$S_i = \{k \mid \text{server } k \text{ can serve class } i\}$

for subset of job classes A: $S(A) = \bigcup_{i \in A} S_i$

$$\lambda(A) = \sum_{i \in A} \lambda_i \qquad \mu(A) = \sum_{k \in S(A)} \mu_k$$

$$\lambda = \lambda(\{1, \dots, J\}) \qquad \mu = \mu(\{1, \dots, M\})$$

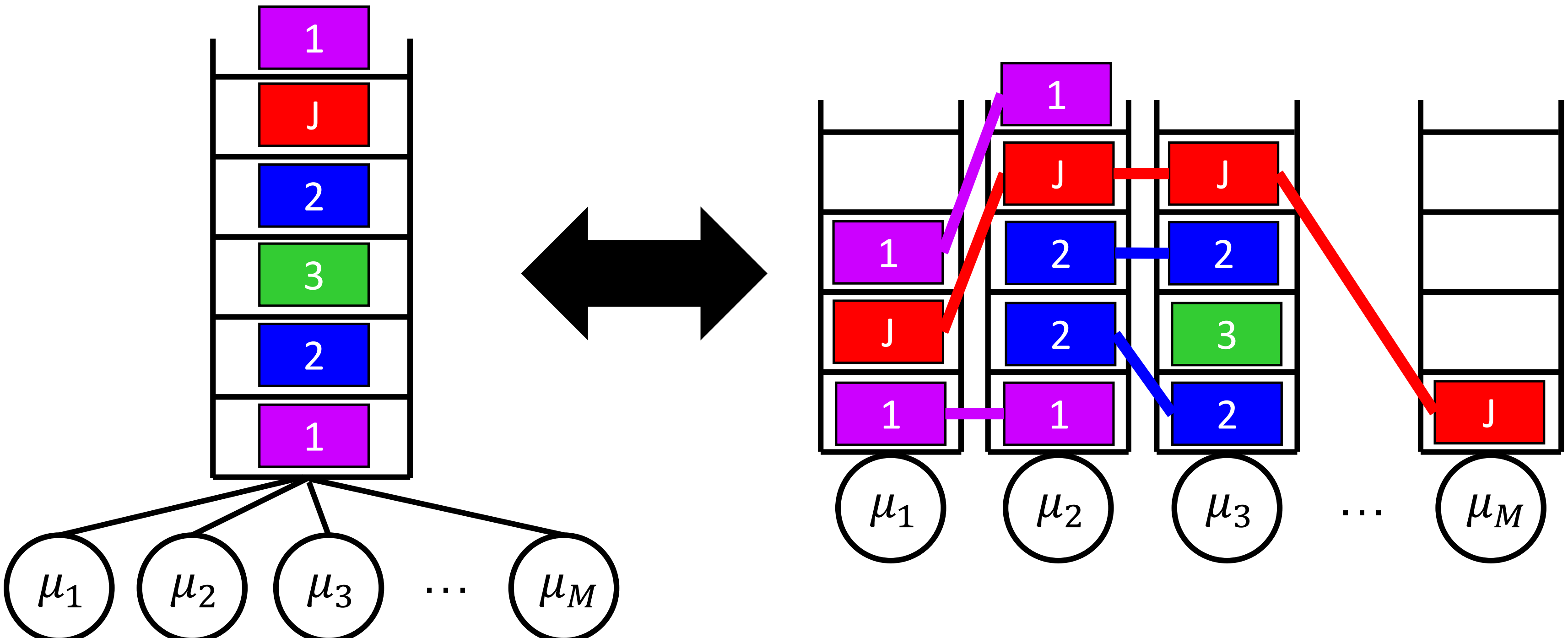
$C_k = \{i \mid \text{server } k \text{ can serve class } i\}$

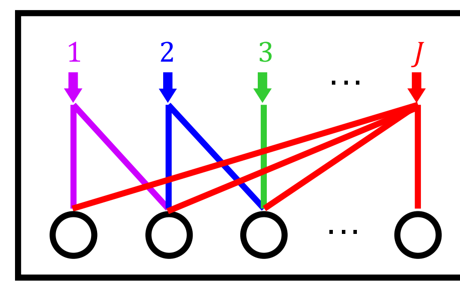
for subset of servers B: $C(B) = \bigcup_{k \in B} C_k$

$$\lambda(B) = \sum_{i \in C(B)} \lambda_i \qquad \mu(B) = \sum_{k \in B} \mu_k$$

Two Equivalent Views

Central Queue | Job Redundancy

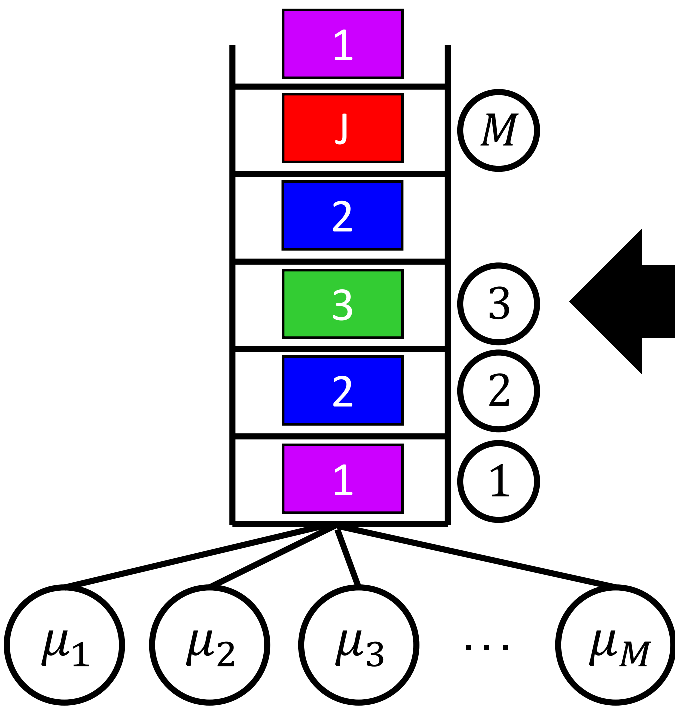




The Noncollaborative Model

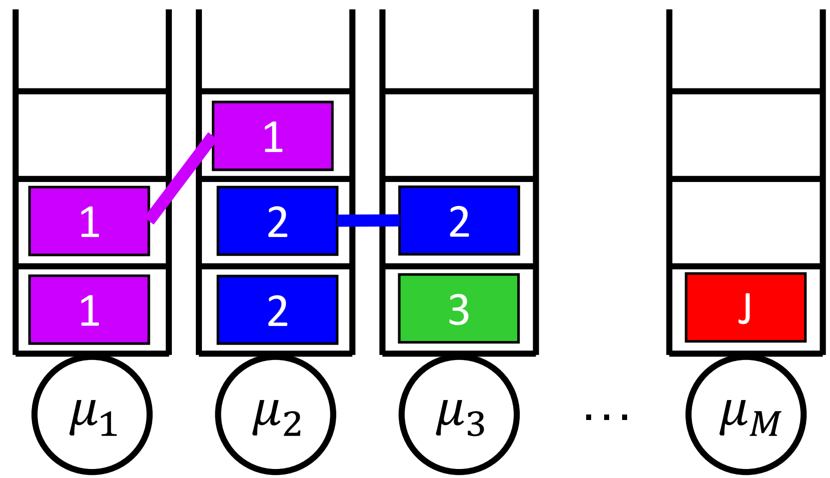
A job can be served by only one server

Central Queue

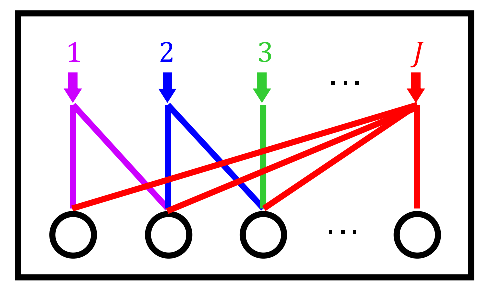


Job Redundancy

"Cancel-on-start" ⟷ Join the smallest work



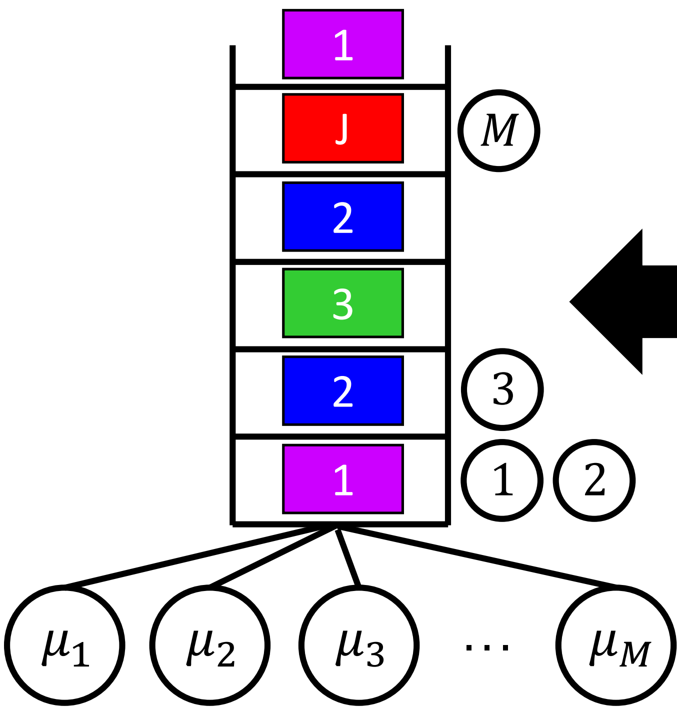
If multiple compatible idle servers upon arrival:
Assign Longest Idle Server (ALIS) [Adan, Weiss]

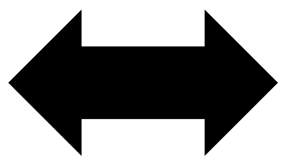


The Collaborative Model

Multiple servers can work on the same job at once, with additive service rate

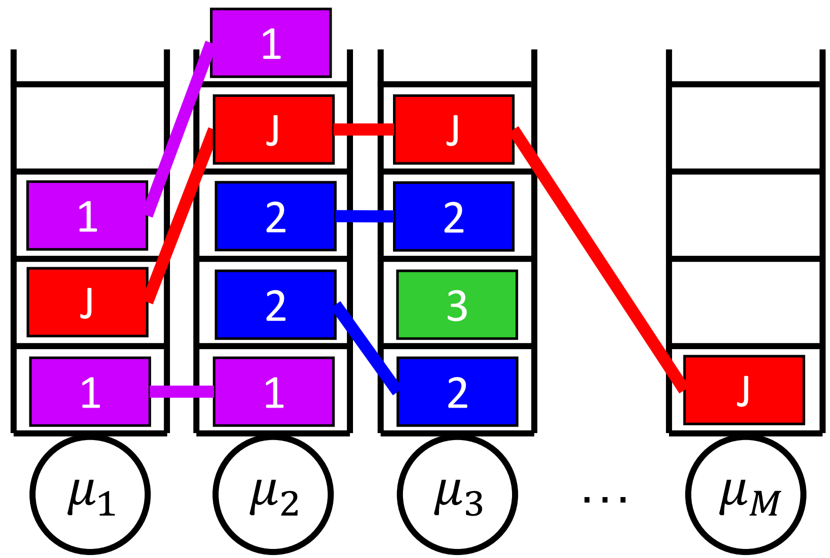
Central Queue



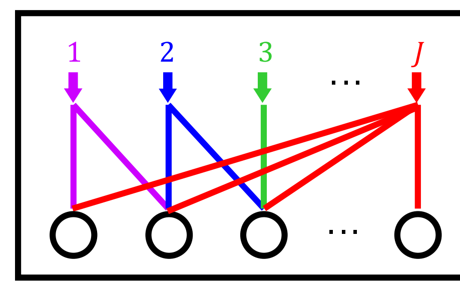


Job Redundancy

"Cancel-on-completion"



If multiple compatible idle servers upon arrival: job enters service on all of them



Outline

- System description
- **Review: reversibility**
- Detailed states

  Collaborative model — Noncollaborative model

  Relating the models
- Partial aggregation

  Collaborative model — Noncollaborative model

  - Special case: fully flexible class
  - Special case: nested systems
- Related product-form systems

Reversibility Review

Def: A stochastic process $X(t)$ is reversible if $(X(t_0), \dots, X(t_n))$ has the same distribution as $(X(t_n), \dots, X(t_0))$ for all $t_0, \dots, t_n$

Reversible ⟺ Detailed balance eqns

$$\pi_i v_{ij} = \pi_j v_{ji}$$

for all states i, j

v_{ij}: rate of going from state i to state j

Def: A queue is quasi-reversible if its state at time t is independent of:

- Arrival times after time t
- Departure times before time t

Quasi-reversible ⟺ Partial balance (for our model)

rate into state i due to arrival = rate out of state i due to departure

rate out of state i due to class-c arrival = rate into state i due to class-c departure

Reversibility Review

Def: A stochastic process $X(t)$ is reversible if $(X(t_0), \dots, X(t_n))$ has the same distribution as $(X(t_n), \dots, X(t_0))$ for all $t_0, \dots, t_n$

Reversible ⟺ Detailed balance eqns

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Def: A queue is quasi-reversible if its state at time t is independent of:

- Arrival times after time t
- Departure times before time t

Quasi-reversible ⟺ Burke's Theorem: Departure process is a Poisson process ⟺ Networks of queues have product-form stationary distribution

Outline

- System description
- Review: reversibility
- **Detailed states**

  Collaborative model Noncollaborative model

  Relating the models

- Partial aggregation

  Collaborative model Noncollaborative model

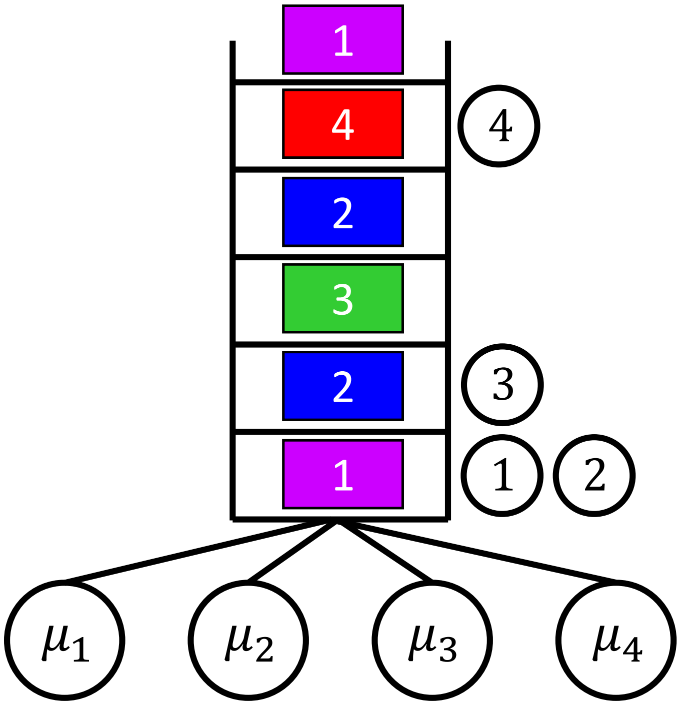
  - Special case: fully flexible class
  - Special case: nested systems

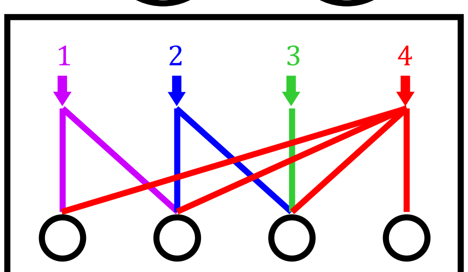
- Related product-form systems

Detailed States: Collaborative Model

$$\text{state } \vec{c}_n = (c_1, c_2, \dots, c_i, \dots, c_n)$$

class of job in position i in the system (including jobs in service)





Total rate of service given to jobs $1, \dots, i$:

$$\mu(\vec{c}_i) = \sum_{k \in S(\vec{c}_i)} \mu_k = \mu(\{c_1, \dots, c_i\})$$

Rate of service given to job i:

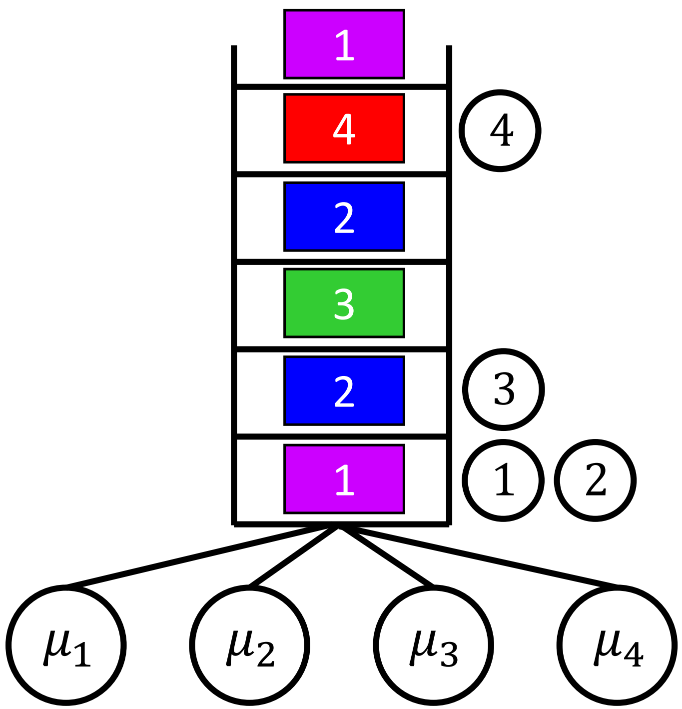
$$\Delta_i(\vec{c}_n) = \sum_{k \in S(\vec{c}_i)} \mu_k - \sum_{k \in S(\vec{c}_{i-1})} \mu_k$$

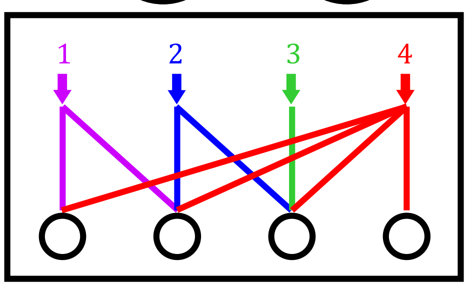
$$= \mu(\vec{c}_i) - \mu(\vec{c}_{i-1})$$

Detailed States: Collaborative Model

$$\text{state } \vec{c}_n = (c_1, c_2, \dots, c_i, \dots, c_n)$$

class of job in position i in the system (including jobs in service)





$$\mu(\vec{c}_i) = \sum_{k \in S(\vec{c}_i)} \mu_k = \mu(\{c_1, \dots, c_i\})$$

$$\Delta_i(\vec{c}_n) = \mu(\vec{c}_i) - \mu(\vec{c}_{i-1})$$

ex. state $\vec{c}_5 = (1,2,3,2,4,1)$

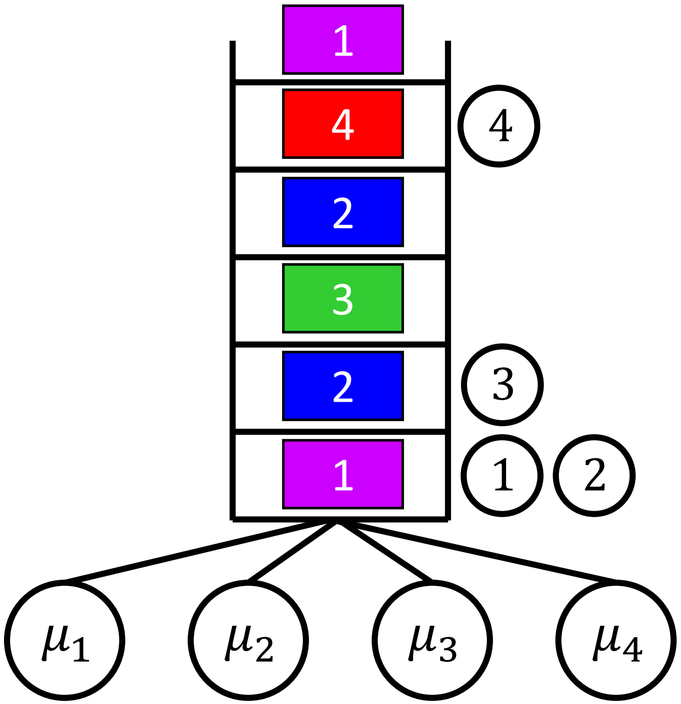
$$\mu(\vec{c}_4) = \mu_1 + \mu_2 + \mu_3$$

$$\Delta_2(\vec{c}_6) = \mu(\vec{c}_2) - \mu(\vec{c}_1)$$

$$= (\mu_1 + \mu_2 + \mu_3) - (\mu_1 + \mu_2) = \mu_3$$

Detailed States: Collaborative Model

state $\vec{c}_n = (c_1, c_2, \ldots, c_i, \ldots, c_n)$

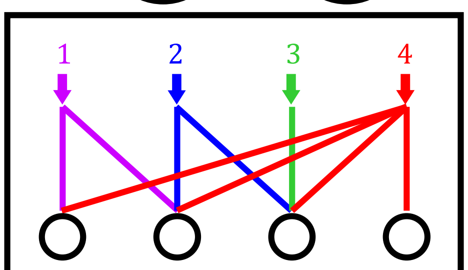


Properties

(i) $\Delta_i(\vec{c}_n) = \Delta_i(\vec{c}_i)$ for all $i \leq n$

(ii) $\mu(\vec{c}_n)$ is the same for any permutation of $c_1, \ldots, c_n$

(iii) $\mu(c) > 0$ for all job classes c



Queues satisfying (i)-(iii) are **order independent** (OI)

OI properties are sufficient to ensure product form stationary distribution

[Krzesinski]

Order Independence

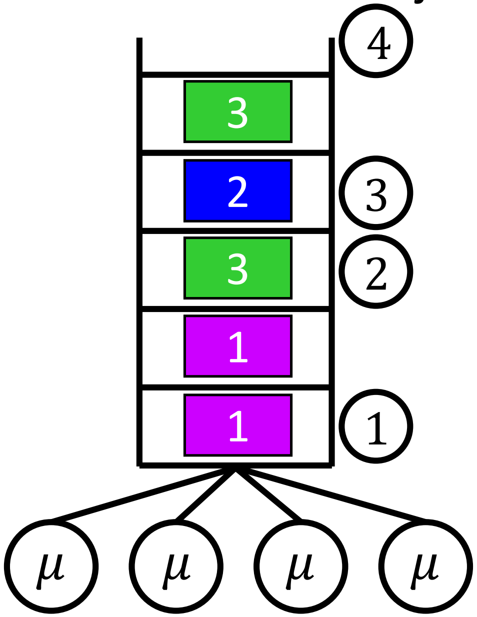
Queues satisfying (i)-(iii) are **order independent** (OI)

OI properties are sufficient to ensure product form stationary distribution

Much more general than collaborative model

ex. Multi-Server Station with Concurrent Classes of Customers (**MSCCC**) queue

- All job classes are compatible with all servers
- At most one job per class can be in service at a time



The MSCCC queue satisfies OI properties ➡ Product-form stationary distribution

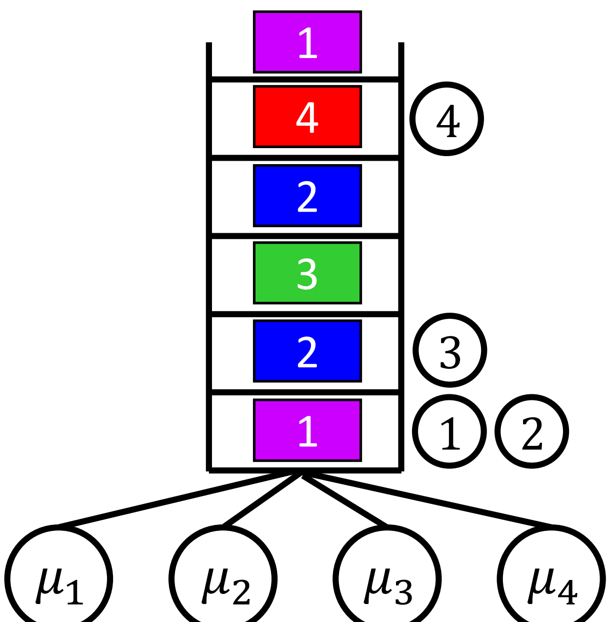
But does not fit within the collaborative model

Detailed States: Collaborative Model

Thm: Given OI properties, the system is quasi-reversible and the stationary distribution is:

$$\pi(\vec{c}_n) = \pi(0) \prod_{i=1}^{n} \frac{\lambda_{c_i}}{\mu(\vec{c}_i)} = \frac{\lambda_{c_n}}{\mu(\vec{c}_n)} \pi(\vec{c}_{n-1})$$

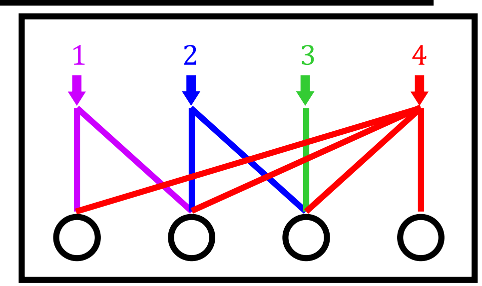
provided $G = \sum_{\vec{c}_n} \prod_{i=1}^{n} \frac{\lambda_{c_i}}{\mu(\vec{c}_i)} < \infty$. Then $\pi(0) = \frac{1}{G}$ is the probability the system is empty.



ex. state $\vec{c}_6 = (1,2,3,2,4,1)$

$$\pi(\vec{c}_6) = \pi(0) \left(\frac{\lambda_1}{\mu_{1,2}}\right)\left(\frac{\lambda_2}{\mu_{1,2,3}}\right)\left(\frac{\lambda_3}{\mu_{1,2,3}}\right)\left(\frac{\lambda_2}{\mu_{1,2,3}}\right)\left(\frac{\lambda_4}{\mu}\right)\left(\frac{\lambda_1}{\mu}\right)$$

[Gardner, Zbarsky, Doroudi, Harchol-Balter, Hyytiä, Scheller-Wolf, Krzesinski, Bonald, Comte]



Detailed States: Collaborative Model

Thm: $$\pi(\vec{c}_n) = \pi(0)\prod_{i=1}^{n}\frac{\lambda_{c_i}}{\mu(\vec{c}_i)} = \frac{\lambda_{c_n}}{\mu(\vec{c}_n)}\pi(\vec{c}_{n-1})$$ ★

[Krzesinski]

Proof: Show ★ satisfies partial balance

rate out of $\vec{c}_n$ due to departure = rate into $\vec{c}_n$ due to arrival

$$\pi(\vec{c}_n)\mu(\vec{c}_n) = \pi(\vec{c}_{n-1})\lambda_{c_n}$$ ✓

Detailed States: Collaborative Model

Thm: $$\pi(\vec{c}_n) = \pi(0)\prod_{i=1}^{n}\frac{\lambda_{c_i}}{\mu(\vec{c}_i)} = \frac{\lambda_{c_n}}{\mu(\vec{c}_n)}\pi(\vec{c}_{n-1}) \quad ★$$

[Krzesinski]

Proof: Show ★ satisfies partial balance

rate out of $\vec{c}_n$ due to class c arrival = rate into $\vec{c}_n$ due to class c departure

Approach: by induction

$$\pi(\vec{c}_n)\lambda_c \overset{?}{=} \sum_{j=1}^{n}\pi(c_1,\ldots,c_j,c,c_{j+1},\ldots,c_n)\Delta_{j+1}(c_1,\ldots,c_j,c,c_{j+1},\ldots,c_n)$$

$$= \sum_{j=1}^{n}\pi(c_1,\ldots,c_j,c,c_{j+1},\ldots,c_n)\Delta_{j+1}(\vec{c}_j,c)$$ (OI condition (i))

Base case ($n = 0$): $\pi(0)\lambda_c = \pi(c)\Delta_1(c) = \pi(c)\mu_c$ ✓

Detailed States: Collaborative Model

Thm:

$$\pi(\vec{c}_n) = \pi(0)\prod_{i=1}^{n}\frac{\lambda_{c_i}}{\mu(\vec{c}_i)} = \frac{\lambda_{c_n}}{\mu(\vec{c}_n)}\pi(\vec{c}_{n-1}) \quad \star$$

[Krzesinski]

Proof: Show ★ satisfies partial balance

Inductive hypothesis: $\pi(\vec{c}_{n-1})\lambda_c = \sum_{j=1}^{n-1}\pi(c_1,\ldots,c_j,c,c_{j+1},\ldots,c_{n-1})\Delta_{j+1}(\vec{c}_j,c)$

$$\pi(\vec{c}_n)\lambda_c \overset{?}{=} \sum_{j=1}^{n}\pi(c_1,\ldots,c_j,c,c_{j+1},\ldots,c_n)\Delta_{j+1}(\vec{c}_j,c)$$

$$= \sum_{j=1}^{n-1}\pi(c_1,\ldots,c_j,c,c_{j+1},\ldots,c_n)\Delta_{j+1}(\vec{c}_j,c) + \pi(\vec{c}_n,c)\Delta_{j+1}(\vec{c}_n,c)$$

Detailed States: Collaborative Model

Thm: $$\pi(\vec{c}_n) = \pi(0) \prod_{i=1}^{n} \frac{\lambda_{c_i}}{\mu(\vec{c}_i)} = \frac{\lambda_{c_n}}{\mu(\vec{c}_n)} \pi(\vec{c}_{n-1})$$ ★

[Krzesinski]

Proof: Show ★ satisfies partial balance

Inductive hypothesis: $$\pi(\vec{c}_{n-1})\lambda_c = \sum_{j=1}^{n-1} \pi\big(c_1, \dots, c_j, c, c_{j+1}, \dots, c_{n-1}\big)\Delta_{j+1}\big(\vec{c}_j, c\big)$$

$$\pi(\vec{c}_n)\lambda_c \stackrel{?}{=} \sum_{j=1}^{n-1} \pi\big(c_1, \dots, c_j, c, c_{j+1}, \dots, c_n\big)\Delta_{j+1}\big(\vec{c}_j, c\big) + \pi(\vec{c}_n, c)\Delta_{j+1}(\vec{c}_n, c)$$

$$= \frac{\lambda_{c_n}}{\mu(\vec{c}_n, c)} \sum_{j=1}^{n-1} \pi\big(c_1, \dots, c_j, c, c_{j+1}, \dots, c_{n-1}\big)\Delta_{j+1}\big(\vec{c}_j, c\big) + \frac{\lambda_c}{\mu(\vec{c}_n, c)} \pi(\vec{c}_n)\Delta_{j+1}(\vec{c}_n, c)$$ (★)

$$= \frac{\lambda_{c_n}}{\mu(\vec{c}_n, c)} \pi(\vec{c}_{n-1})\lambda_c + \frac{\lambda_c}{\mu(\vec{c}_n, c)} \pi(\vec{c}_n)\big(\mu(\vec{c}_n, c) - \mu(\vec{c}_n)\big)$$ (Inductive hypothesis)
(def. of $\Delta_{j+1}(\vec{c}_n, c)$)

Detailed States: Collaborative Model

Thm: $$\pi(\vec{c}_n) = \pi(0)\prod_{i=1}^{n}\frac{\lambda_{c_i}}{\mu(\vec{c}_i)} = \frac{\lambda_{c_n}}{\mu(\vec{c}_n)}\pi(\vec{c}_{n-1}) \quad ★$$

[Krzesinski]

Proof: Show ★ satisfies partial balance

Inductive hypothesis: $$\pi(\vec{c}_{n-1})\lambda_c = \sum_{j=1}^{n-1}\pi\left(c_1,\dots,c_j,c,c_{j+1},\dots,c_{n-1}\right)\Delta_{j+1}\left(\vec{c}_j,c\right)$$

$$\pi(\vec{c}_n)\lambda_c \overset{?}{=} \sum_{j=1}^{n-1}\pi\left(c_1,\dots,c_j,c,c_{j+1},\dots,c_n\right)\Delta_{j+1}\left(\vec{c}_j,c\right) + \pi(\vec{c}_n,c)\Delta_{j+1}(\vec{c}_n,c)$$

$$= \frac{\lambda_{c_n}}{\mu(\vec{c}_n,c)}\sum_{j=1}^{n-1}\pi\left(c_1,\dots,c_j,c,c_{j+1},\dots,c_{n-1}\right)\Delta_{j+1}\left(\vec{c}_j,c\right) + \frac{\lambda_c}{\mu(\vec{c}_n,c)}\pi(\vec{c}_n)\Delta_{j+1}(\vec{c}_n,c) \quad (★)$$

$$= \frac{\lambda_{c_n}}{\mu(\vec{c}_n,c)}\pi(\vec{c}_{n-1})\lambda_c + \frac{\lambda_c}{\mu(\vec{c}_n,c)}\pi(\vec{c}_n)\left(\mu(\vec{c}_n,c) - \mu(\vec{c}_n)\right)$$

(Inductive hypothesis)
(def. of $\Delta_{j+1}(\vec{c}_n,c)$)

Detailed States: Collaborative Model

Thm: $$\pi(\vec{c}_n) = \pi(0)\prod_{i=1}^{n}\frac{\lambda_{c_i}}{\mu(\vec{c}_i)} = \frac{\lambda_{c_n}}{\mu(\vec{c}_n)}\pi(\vec{c}_{n-1}) \quad ★$$

[Krzesinski]

Proof: Show ★ satisfies partial balance

Inductive hypothesis: $$\pi(\vec{c}_{n-1})\lambda_c = \sum_{j=1}^{n-1}\pi(c_1,\ldots,c_j,c,c_{j+1},\ldots,c_{n-1})\Delta_{j+1}(\vec{c}_j,c)$$

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$$= \frac{\lambda_{c_n}}{\mu(\vec{c}_n,c)}\sum_{j=1}^{n-1}\pi(c_1,\ldots,c_j,c,c_{j+1},\ldots,c_{n-1})\Delta_{j+1}(\vec{c}_j,c) + \frac{\lambda_c}{\mu(\vec{c}_n,c)}\pi(\vec{c}_n)\Delta_{j+1}(\vec{c}_n,c) \quad (★)$$

$$= \lambda_c\pi(\vec{c}_n)$$ ✓

(just a little algebra)

Computations for Collaborative Model

Thm: $\pi(0)$ can be computed recursively:

$$\pi(0) = \frac{\mu - \lambda}{\sum_{s \in M} \frac{\mu_s}{\pi_{-s}(0)}}$$

where -s denotes a system in which server s and all job classes compatible with server s are removed

Thm: Let $E\left[N^{(i)}\right]$ and $E[N]$ denote the number of class i jobs and the total number of jobs in the system.
$E\left[N^{(i)}\right]$ and $E[N]$ can be computed recursively:

$$E\left[N^{(i)}\right] = \frac{\lambda_i + \sum_{s \in M \setminus C_i} \mu_s \pi^{(s)}(0) E\left[N_{-s}^{(i)}\right]}{\mu - \lambda}$$

$$E[N] = \frac{\lambda + \sum_{s \in M} \mu_s \pi^{(s)}(0) N_{-s}}{\mu - \lambda}$$

[Bonald, Comte, Mathieu]

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- System description
- Review: reversibility
- **Detailed states**

  Collaborative model **Noncollaborative model**

  Relating the models
- Partial aggregation

  Collaborative model Noncollaborative model
  - Special case: fully flexible class
  - Special case: nested systems
- Related product-form systems

Detailed States: Noncollaborative Model

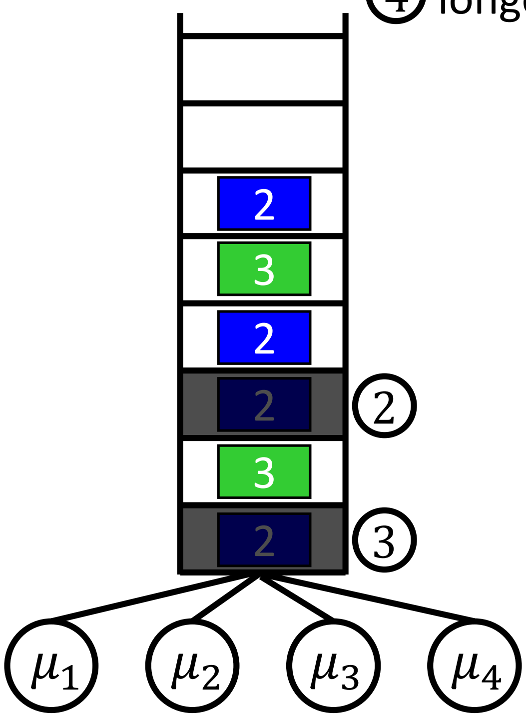
$$\text{state } (\vec{c}_n, \vec{s}_k) = (c_1, \dots, c_i, \dots, c_n; s_1, \dots, s_l, \dots, s_k)$$

class of job in position i in the **queue** (not including jobs in service)

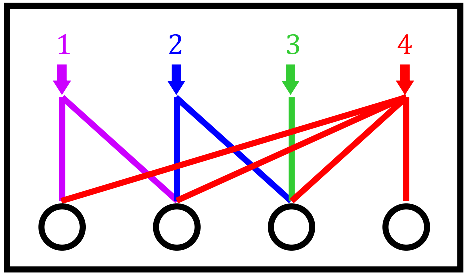
server that has been idle lth longest

① most recently idle

④ longest idle



Recall: Assign Longest Idle Server (ALIS)
if arrival finds multiple compatible idle servers

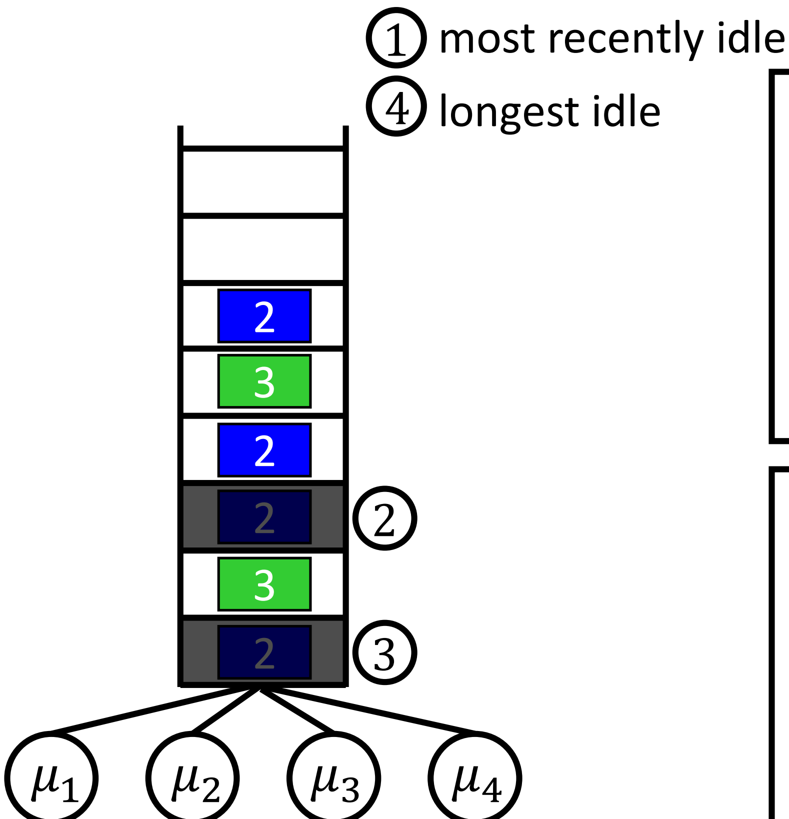


Detailed States: Noncollaborative Model

state $(\vec{c}_n, \vec{s}_k) = (c_1, \ldots, c_i, \ldots, c_n; s_1, \ldots, s_l, \ldots, s_k)$

class of job in position i in the **queue** (not including jobs in service)

server that has been idle lth longest



$$\mu(\vec{c}_i) = \sum_{k \in S(\vec{c}_i)} \mu_k = \mu(\{c_1, \ldots, c_i\})$$

$$\Delta_i(\vec{c}_n) = \mu(\vec{c}_i) - \mu(\vec{c}_{i-1}) =$$ rate at which job i leaves the **queue** (enters service)

ex. state $(\vec{c}_4; \vec{s}_2) = (3,2,3,2; 4,1)$

$$\mu(\vec{c}_4) = \mu_2 + \mu_3$$

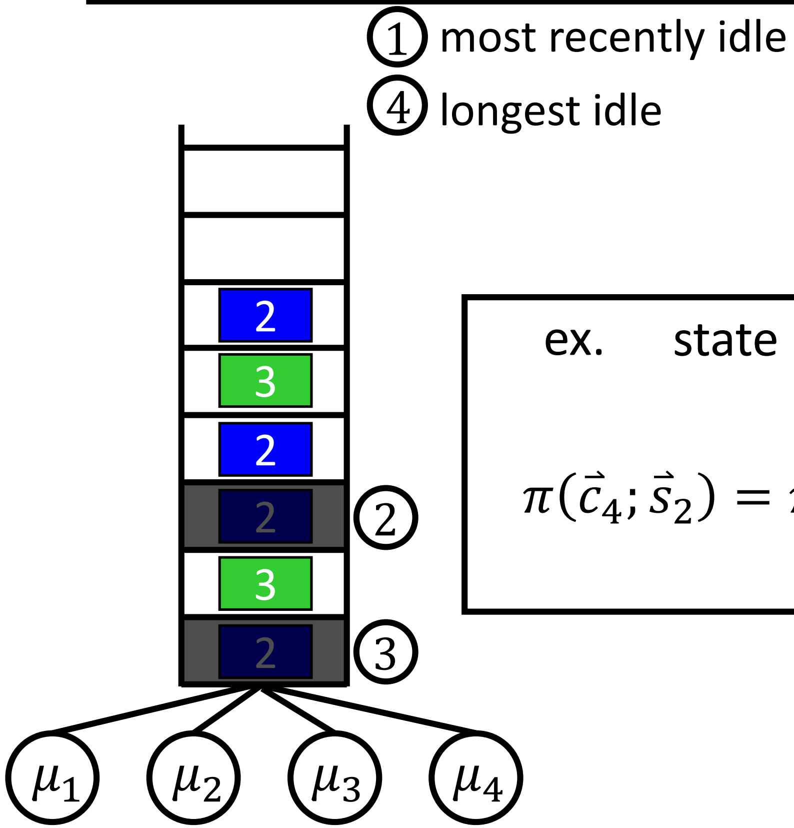
$$\Delta_2(\vec{c}_4) = \mu(\vec{c}_2) - \mu(\vec{c}_1) = (\mu_2 + \mu_3) - \mu_3 = \mu_2$$

Detailed States: Noncollaborative Model

Thm:

$$\pi(\vec{c}_n; \vec{s}_k) = \pi(0;0) \prod_{i=1}^{n} \frac{\lambda_{c_i}}{\mu(\vec{c}_i)} \prod_{l=1}^{k} \frac{\mu_{s_l}}{\lambda(\vec{s}_l)}$$

where $\pi(0;0)$ is the probability that the queue is empty and there are no idle servers



$\pi(0;0) \neq \Pr\{\text{system is empty}\}$

$\pi(0;0) \neq \pi(0)$ in collab system

ex. state $(\vec{c}_4; \vec{s}_2) = (3,2,3,2; 4,1)$

$$\pi(\vec{c}_4; \vec{s}_2) = \pi(0;0) \left(\frac{\lambda_3}{\mu_3}\right) \left(\frac{\lambda_2}{\mu_{2,3}}\right) \left(\frac{\lambda_3}{\mu_{2,3}}\right) \left(\frac{\lambda_2}{\mu_{2,3}}\right) \left(\frac{\mu_4}{\lambda_4}\right) \left(\frac{\mu_{1,4}}{\lambda_{1,4}}\right)$$

Detailed States: Noncollaborative Model

Thm:

$$\pi(\vec{c}_n; \vec{s}_k) = \pi(0;0) \prod_{i=1}^{n} \frac{\lambda_{c_i}}{\mu(\vec{c}_i)} \prod_{l=1}^{k} \frac{\mu_{s_l}}{\lambda(\vec{s}_l)}$$

where $\pi(0;0)$ is the probability that the queue is empty and there are no idle servers

Proof: Partial balance for job c_i: Similar to collaborative case

Partial balance for server s_l:

Let $\Delta_l^\lambda(\vec{s}_k) = \lambda(\vec{s}_l) - \lambda(\vec{s}_{l-1})$ denote the rate at which the lth idle server will become busy

Observation: idle servers satisfy OI properties:

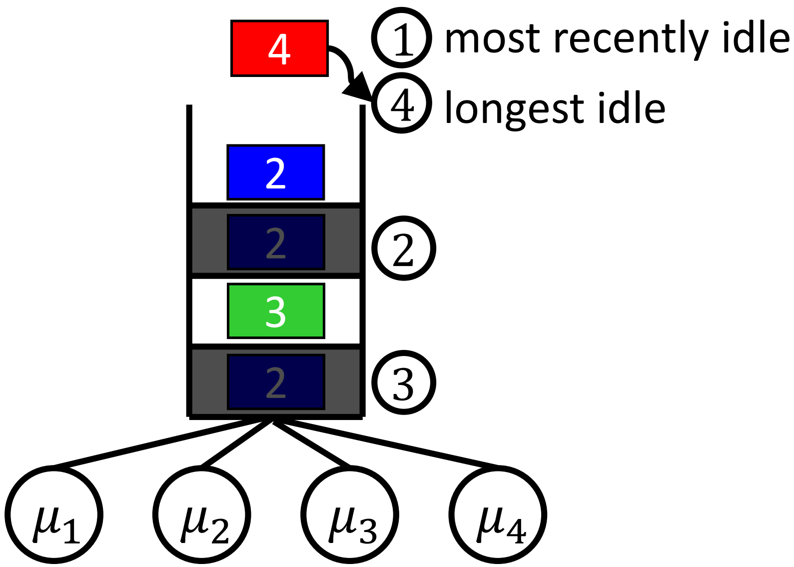
(i) $\Delta_l^\lambda(\vec{s}_k) = \Delta_l^\lambda(\vec{s}_l)$ for all $l \leq k$

(ii) $\lambda(\vec{s}_l)$ is the same for any permutation of $s_1, \dots, s_l$

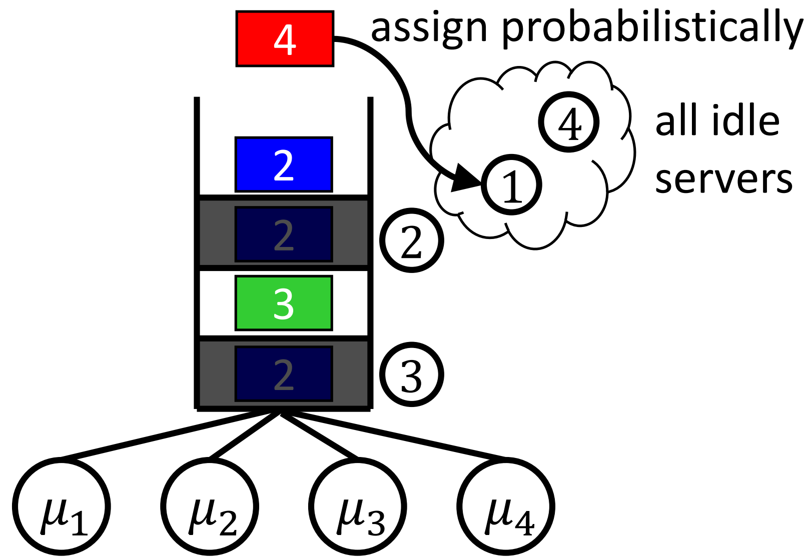
(iii) $\lambda(s) > 0$ for all servers s

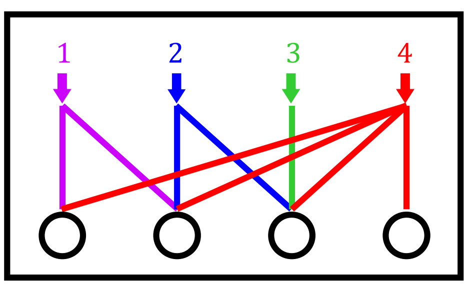
Noncollaborative with Random Assignment

Assign Longest Idle Server



Random Assignment





$\lambda^{A}_{b_l}\left(\vec{b}_{l-1}\right)$: activation rate of server b_l, given busy servers $\vec{b}_{l-1}$

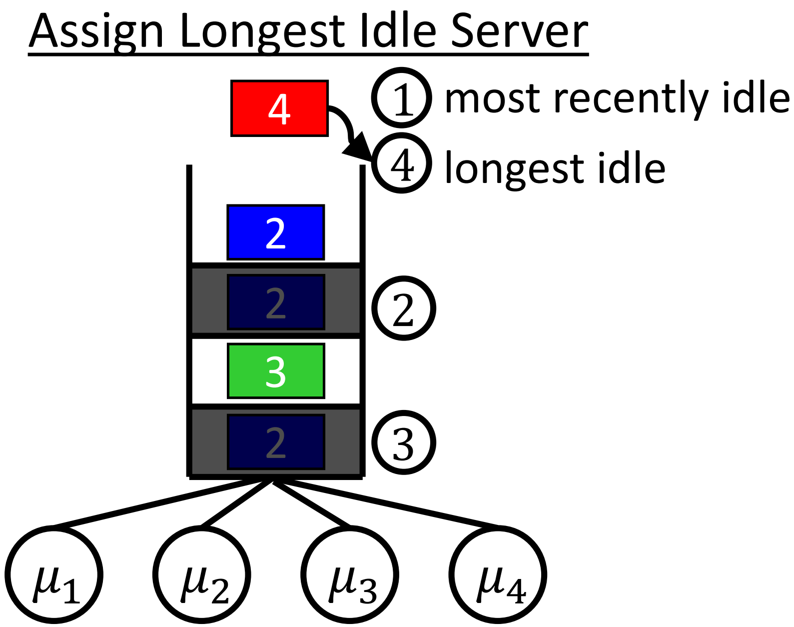
Activation rate condition: For all $1 \le m \le M$,

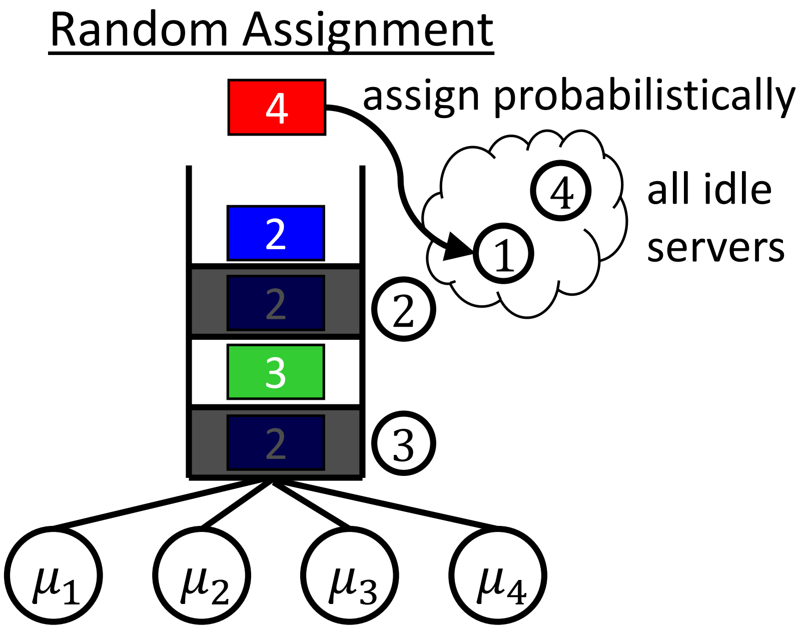
$$\prod_{l=1}^{m} \lambda^{A}_{b_l}\left(\vec{b}_{l-1}\right) \text{ is the same for all permutations of } \vec{b}_m$$

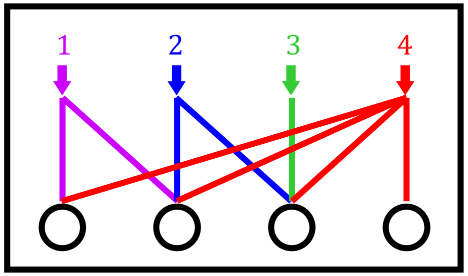
(similar to order independence for busy servers)

[Adan, Visschers, Weiss]

Noncollaborative with Random Assignment







Thm: Given the activation rate condition:

$$\pi^{RAND}\left(\vec{c}_n; \vec{b}_m\right) = \pi^{RAND}(0;0) \prod_{i=1}^{n} \frac{\lambda_{c_i}}{\mu(\vec{c}_i)} \prod_{l=1}^{m} \frac{\lambda_{b_l}^{A}\left(\vec{b}_{l-1}\right)}{\mu\left(\vec{b}_l\right)}$$

[Adan, Visschers, Weiss]

Outline

- System description
- Review: reversibility
- **Detailed states**

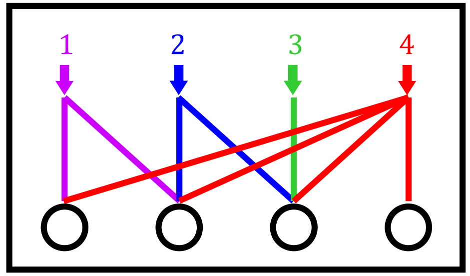
  Collaborative model — Noncollaborative model

  Relating the models
- Partial aggregation

  Collaborative model — Noncollaborative model
  - Special case: fully flexible class
  - Special case: nested systems
- Related product-form systems

Collaborative vs. Noncollaborative

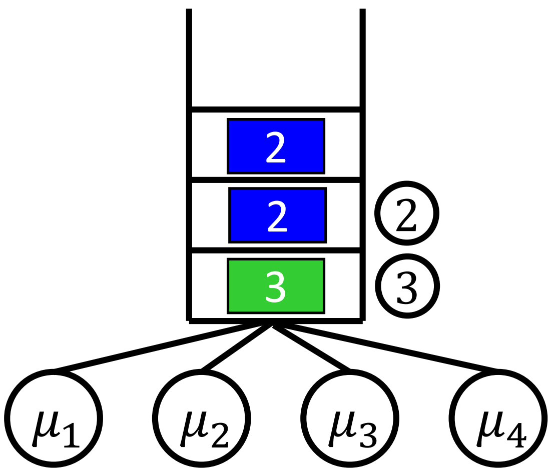
Collaborative (C)

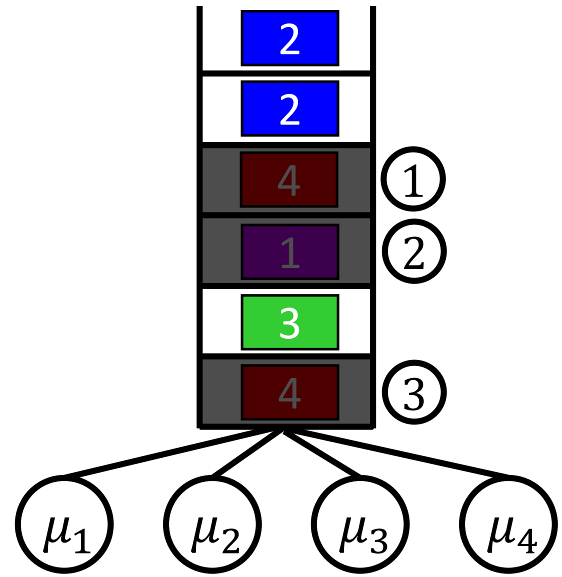


$\vec{c}_n = (3,2,2)$

Noncollaborative (NC) ALIS

$\vec{s}_k = (4)$

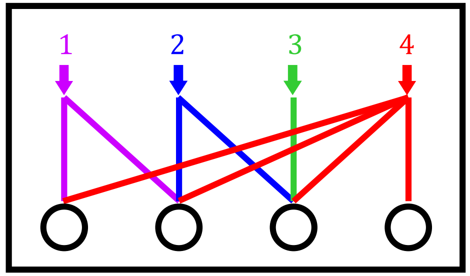




$$\pi(3,2,2) = \pi(0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)$$

$$\pi(3,2,2;4) = \pi(0;0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\mu_4}{\lambda_4}\right)$$

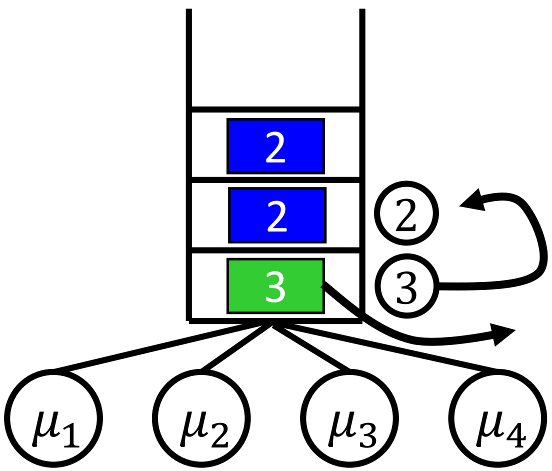
Collaborative vs. Noncollaborative



$$\vec{c}_n = (3,2,2)$$

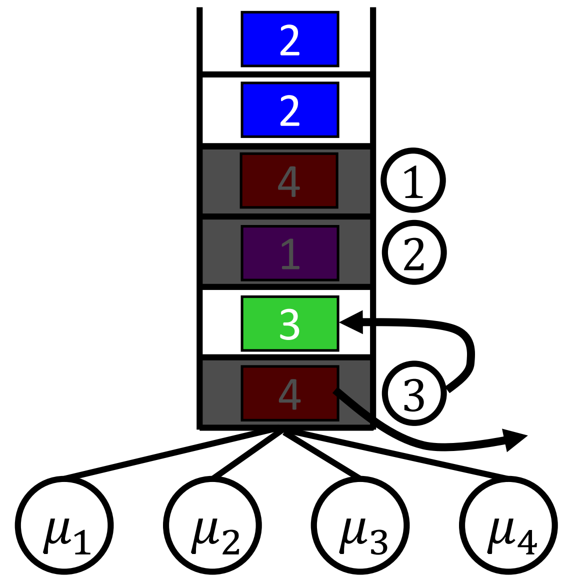
$$\vec{s}_k = (4)$$

Collaborative (C)



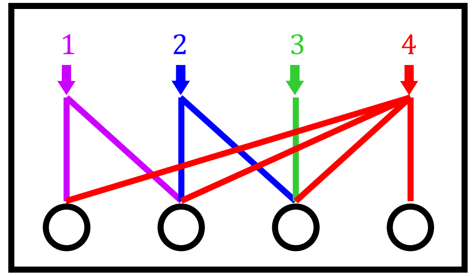
$$\pi(3,2,2) = \pi(0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)$$

Noncollaborative (NC) ALIS



$$\pi(3,2,2;4) = \pi(0;0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\mu_4}{\lambda_4}\right)$$

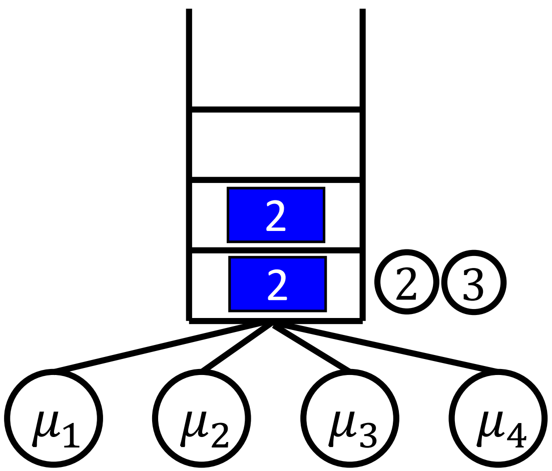
Collaborative vs. Noncollaborative



$$\vec{c}_n = (2,2)$$

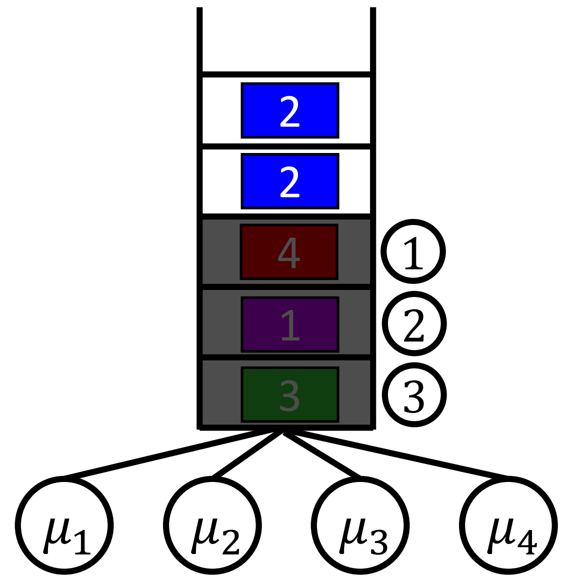
$$\vec{s}_k = (4)$$

Collaborative (C)



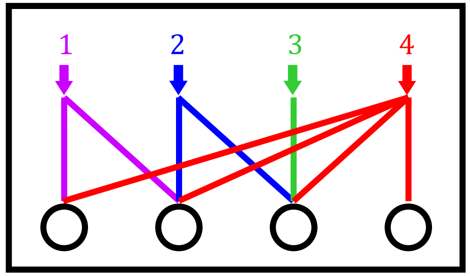
$$\pi(2,2) = \pi(0)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)$$

Noncollaborative (NC) ALIS



$$\pi(2,2;4) = \pi(0;0)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\mu_4}{\lambda_4}\right)$$

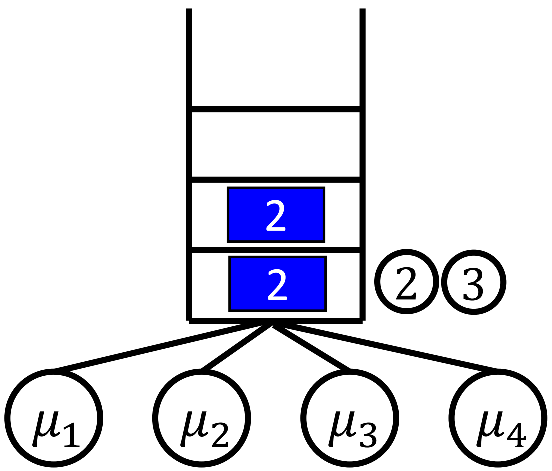
Collaborative vs. Noncollaborative



$\vec{c}_n = (2,2)$

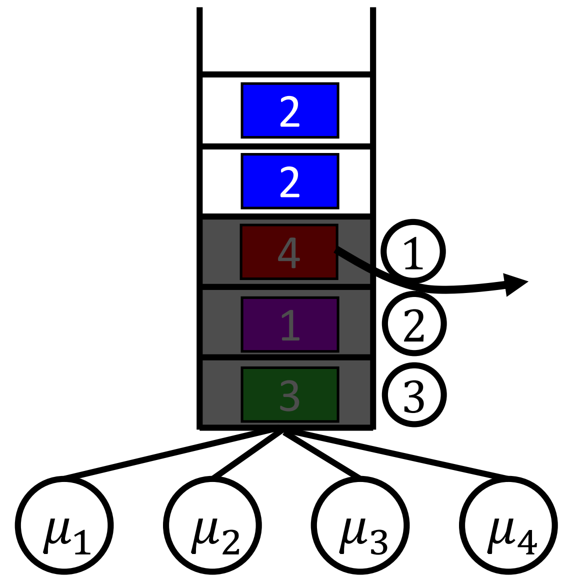
$\vec{s}_k = (4)$

Collaborative (C)



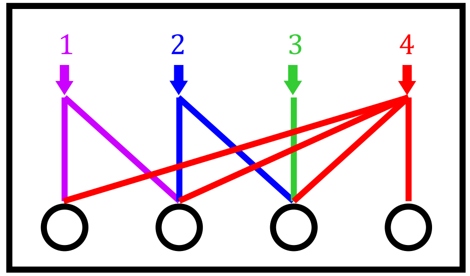
$$\pi(2,2) = \pi(0)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)$$

Noncollaborative (NC) ALIS



$$\pi(2,2;4) = \pi(0;0)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\mu_4}{\lambda_4}\right)$$

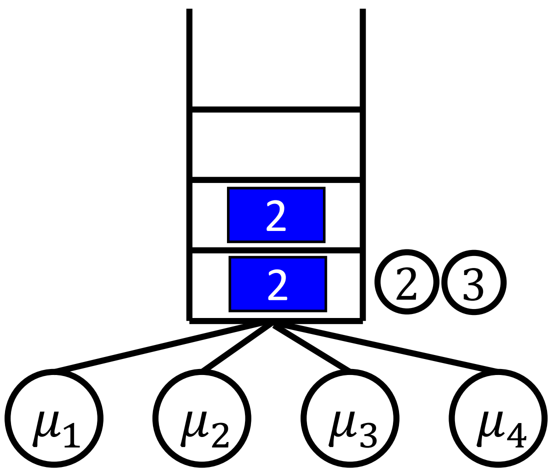
Collaborative vs. Noncollaborative



$\vec{c}_n = (2,2)$

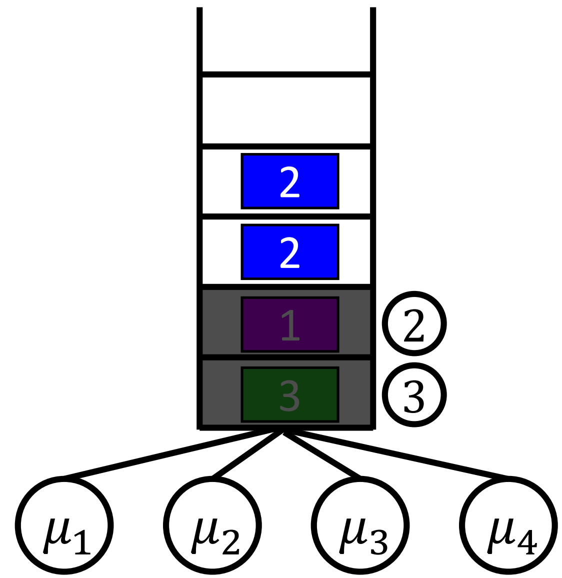
$\vec{s}_k = (4,1)$

Collaborative (C)



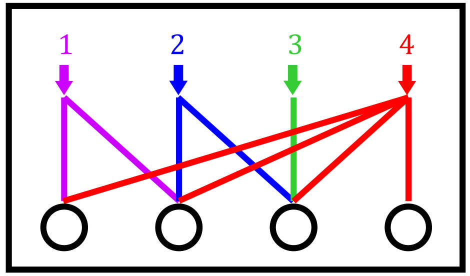
$$\pi(2,2) = \pi(0)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)$$

Noncollaborative (NC) ALIS



$$\pi(2,2;4,1) = \pi(0;0)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\mu_4}{\lambda_4}\right)\left(\frac{\mu_{1,4}}{\lambda_{1,4}}\right)$$

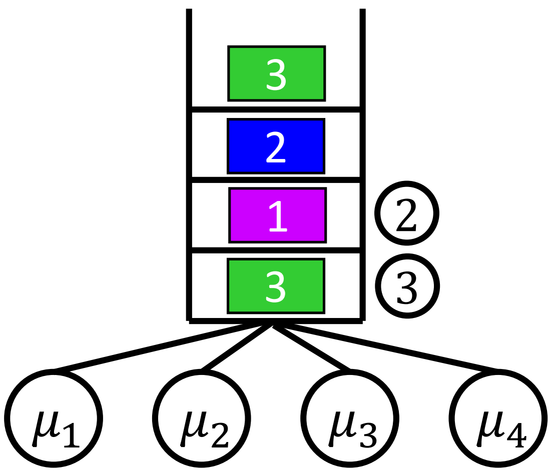
Collaborative vs. Noncollaborative



$\vec{c}_n = (3,1,2,3)$

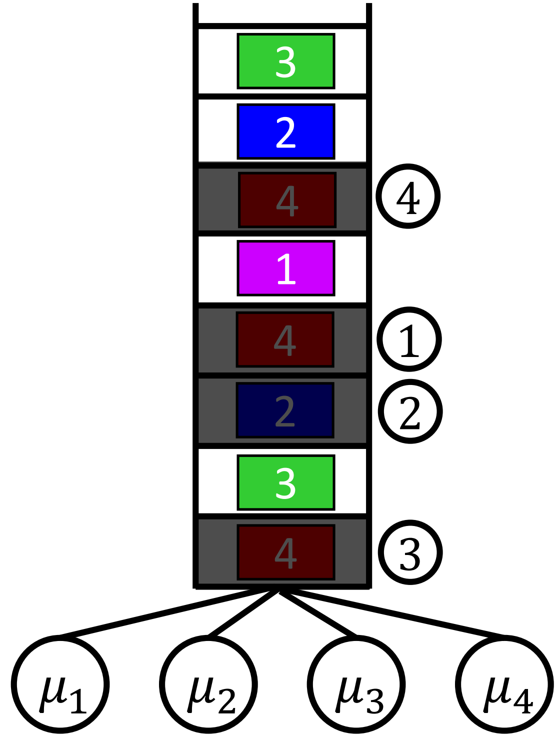
All servers are busy in noncollaborative system

Collaborative (C)



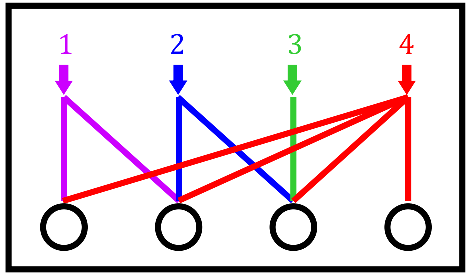
$$\pi(3,1,2,3) = \pi(0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_1}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_3}{\mu_{2,3}}\right)$$

Noncollaborative (NC) ALIS



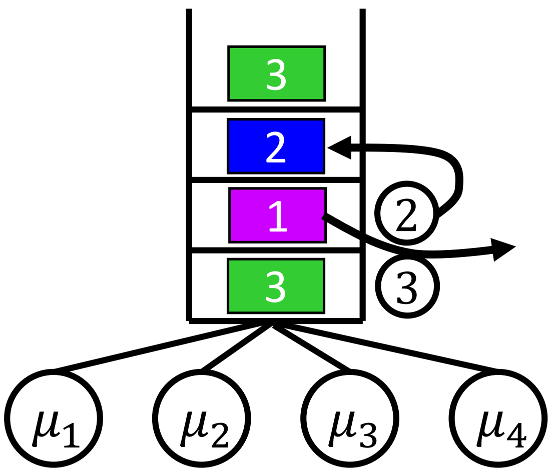
$$\pi(3,1,2,3;0) = \pi(0;0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_1}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_3}{\mu_{2,3}}\right)$$

Collaborative vs. Noncollaborative



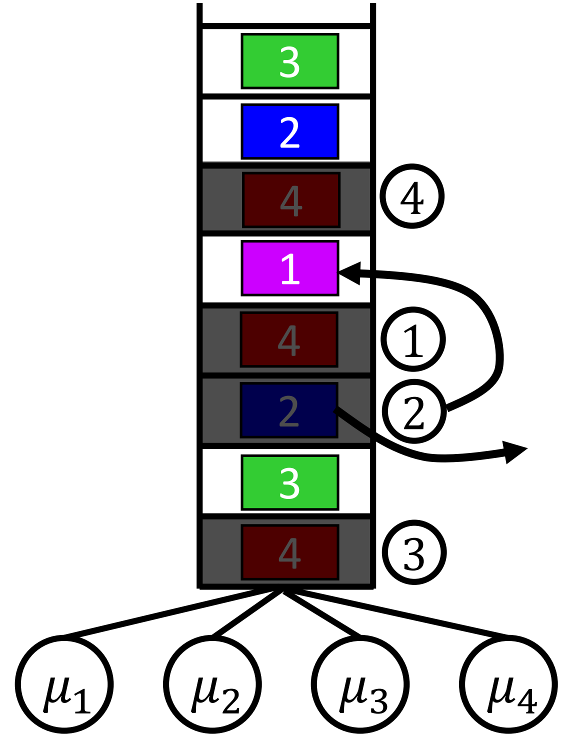
$\vec{c}_n = (3,1,2,3)$

Collaborative (C)



All servers are busy in noncollaborative system

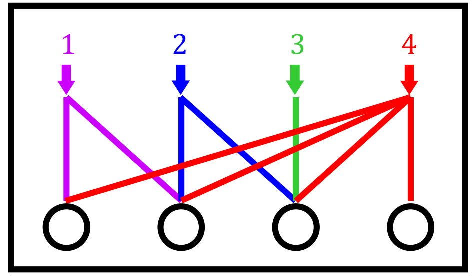
Noncollaborative (NC) ALIS



$$\pi(3,1,2,3) = \pi(0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_1}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_3}{\mu_{2,3}}\right)$$

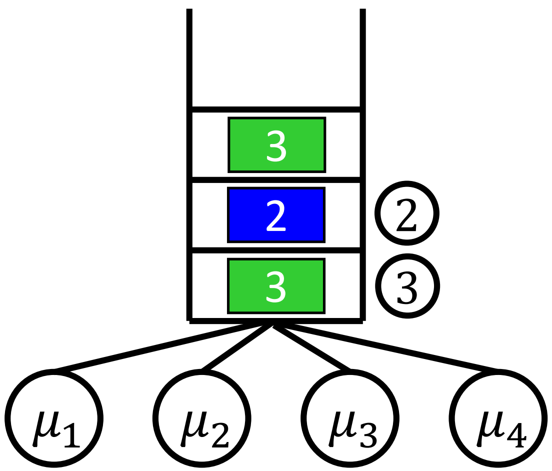
$$\pi(3,1,2,3;0) = \pi(0;0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_1}{\mu_{2,3}}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_3}{\mu_{2,3}}\right)$$

Collaborative vs. Noncollaborative



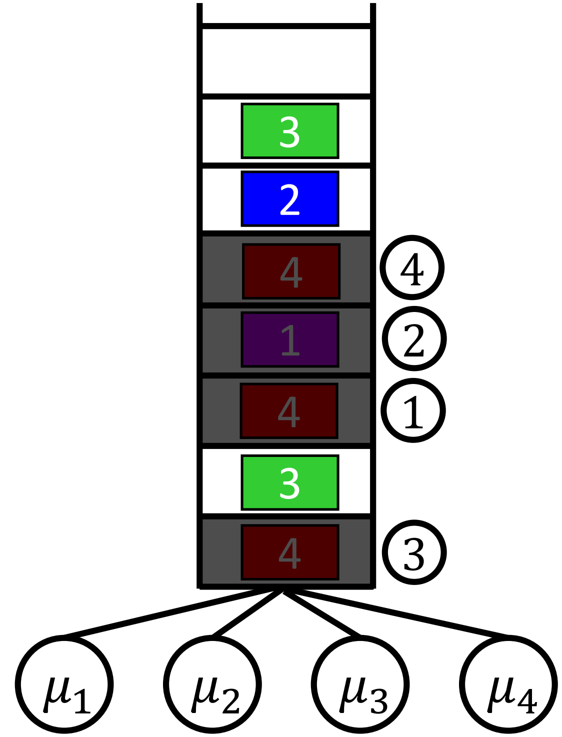
$\vec{c}_n = (3,2,3)$

Collaborative (C)



All servers are busy in noncollaborative system

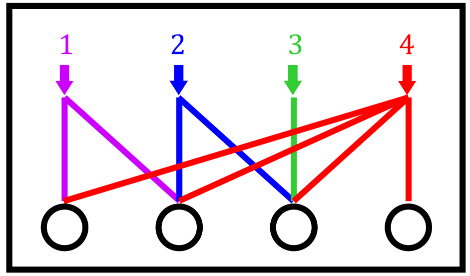
Noncollaborative (NC) ALIS



$$\pi(3,2,3) = \pi(0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_3}{\mu_{2,3}}\right)$$

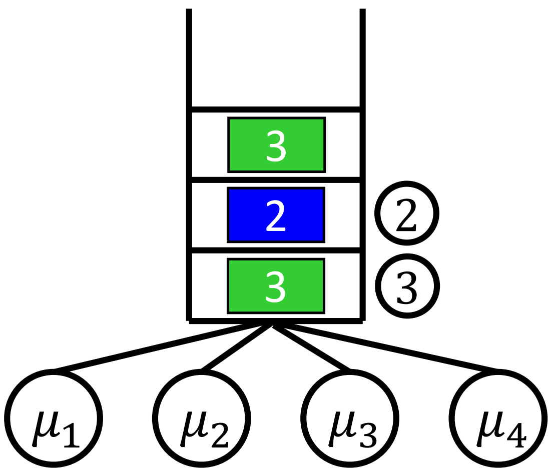
$$\pi(3,2,3;0) = \pi(0;0)\left(\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_2}{\mu_{2,3}}\right)\left(\frac{\lambda_3}{\mu_{2,3}}\right)$$

Collaborative vs. Noncollaborative

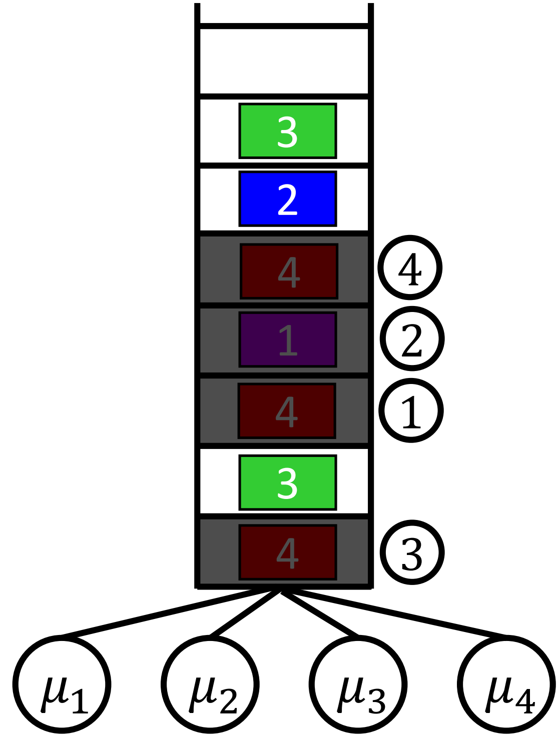


$\vec{c}_n = (3,2,3)$

Collaborative (C)



Noncollaborative (NC) ALIS



Conditioned on all servers being busy in NC, the noncollaborative **queue** has the same stationary distribution as the collaborative **system**

Outline

- System description
- Review: reversibility
- Detailed states

  Collaborative model　　　　Noncollaborative model

  Relating the models

- **Partial aggregation**

  Collaborative model　　　　Noncollaborative model

  - Special case: fully flexible class
  - Special case: nested systems

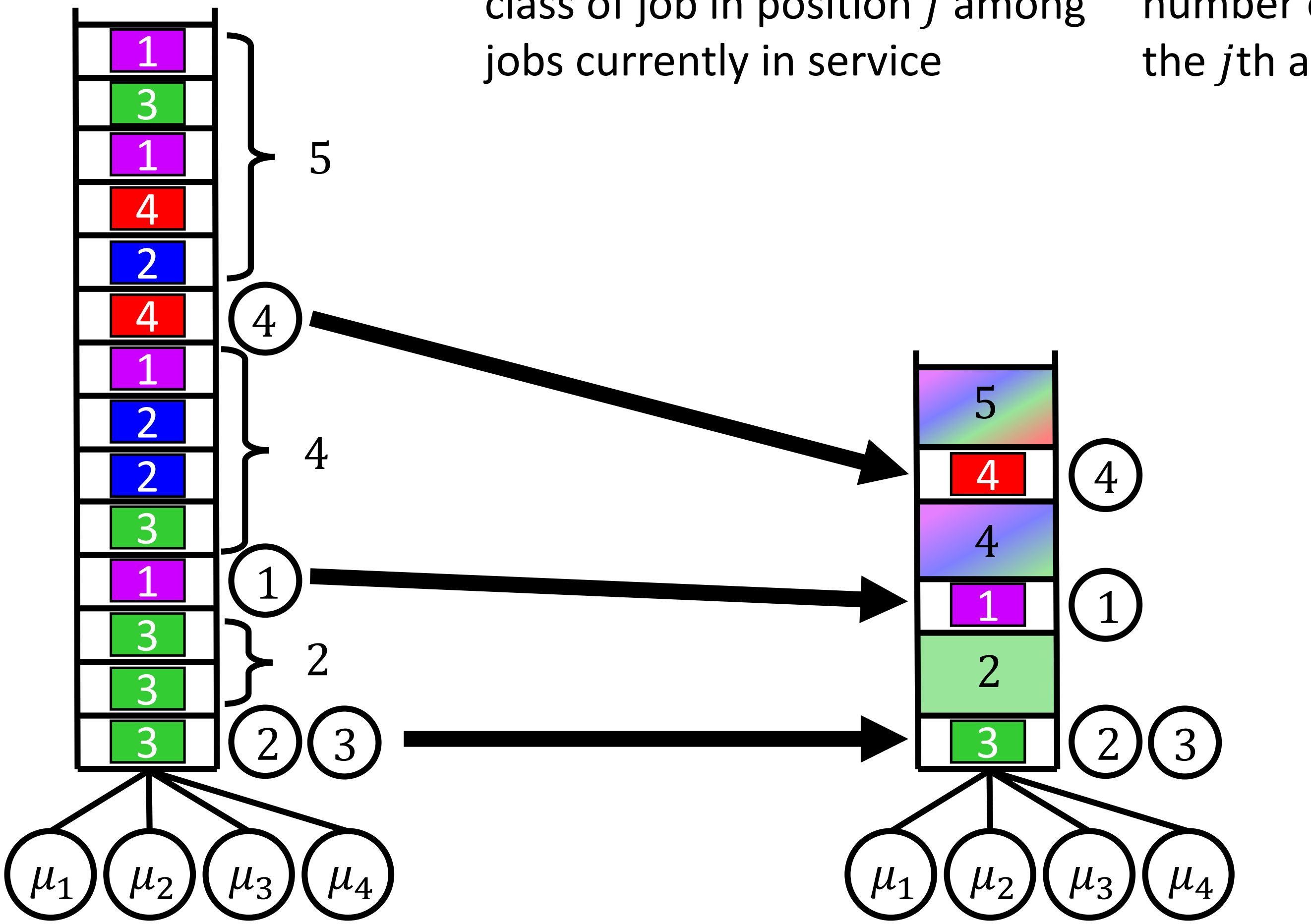
- Related product-form systems

Partial Aggregation: Collaborative Model

$$\text{state } \left(\vec{d}_l;\vec{n}_l\right) = \left(d_1,\dots,d_j,\dots,d_l;n_1,\dots,n_j,\dots,n_l\right)$$

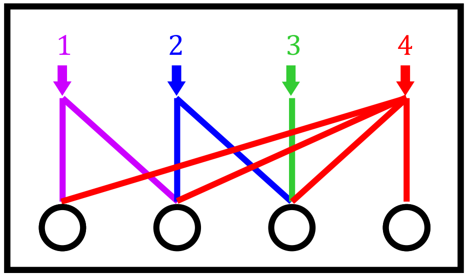
class of job in position j among jobs currently in service

number of jobs in the queue between the jth and $(j+1)$st jobs in service



$\vec{c}_n = (3,3,3,1,3,2,2,1,4,2,4,1,3,1)$

$\left(\vec{d}_l, n_l\right) = (3,1,4;2,4,5)$



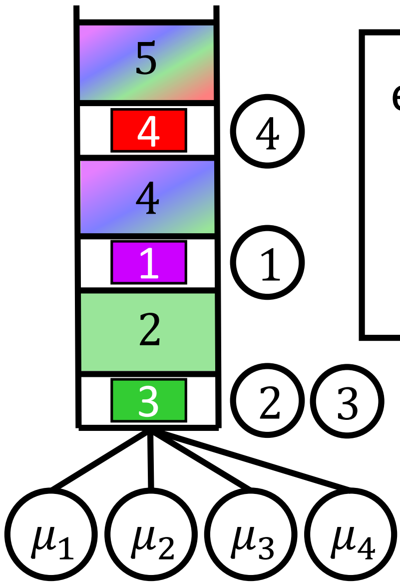
Partial Aggregation: Collaborative Model

Thm:

$$\pi\left(\vec{d}_l, \vec{n}_l\right) = \pi(0) \prod_{j=1}^{l} \frac{\lambda_{d_j}}{\mu\left(\vec{d}_j\right)} \left(\frac{\lambda\left(R\left(\vec{d}_j\right)\right)}{\mu\left(\vec{d}_j\right)} \right)^{n_j}$$

where $\pi(0)$ is the probability that the system is empty

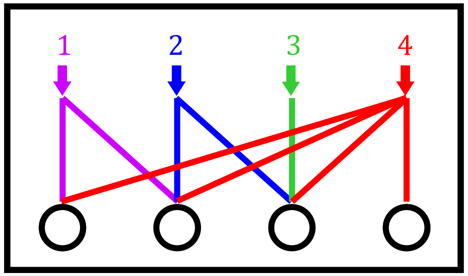
$$R\left(\vec{d}_j\right) = \left\{ i \,\middle|\, \text{class } i \textbf{ R}\text{equires a server in } S\left(\vec{d}_j\right) \right\}$$



$\left(\vec{d}_l, n_l\right) = (3,1,4;2,4,5)$

ex. state $\left(\vec{d}_3, n_3\right) = (3,1,4;2,4,5)$

$$\pi\left(\vec{d}_3, n_3\right) = \pi(0) \left(\frac{\lambda_3}{\mu_{2,3}}\right) \left(\frac{\lambda_3}{\mu_{2,3}}\right)^2 \left(\frac{\lambda_1}{\mu_{1,2,3}}\right) \left(\frac{\lambda_{1,2,3}}{\mu_{1,2,3}}\right)^4 \left(\frac{\lambda_4}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^5$$



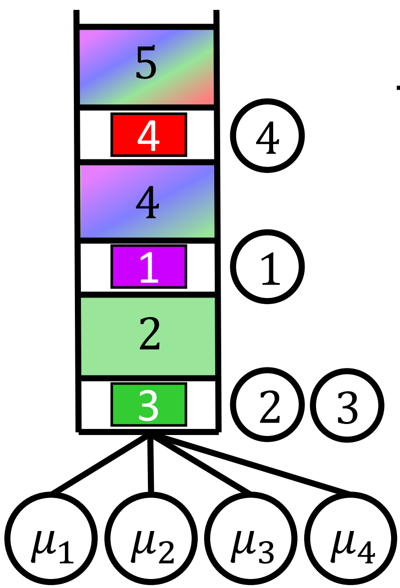
Partial Aggregation: Collaborative Model

Thm:

$$\pi\left(\vec{d}_l, \vec{n}_l\right) = \pi(0) \prod_{j=1}^{l} \frac{\lambda_{d_j}}{\mu\left(\vec{d}_j\right)} \left(\frac{\lambda\left(R\left(\vec{d}_j\right)\right)}{\mu\left(\vec{d}_j\right)} \right)^{n_j}$$

where $\pi(0)$ is the probability that the system is empty

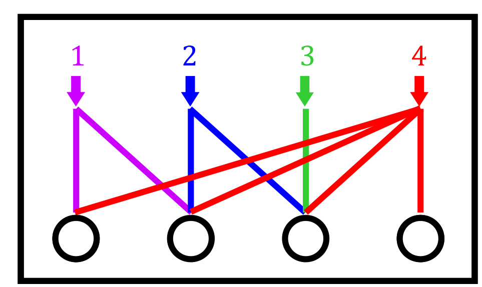
$$R\left(\vec{d}_j\right) = \left\{ i \middle| \text{class } i \textbf{ R}\text{equires a server in } S\left(\vec{d}_j\right) \right\}$$



Proof: Approach 1: Partially aggregated states satisfy partial balance

Approach 2: Sum up detailed states:

$$\pi\left(\vec{d}_l, \vec{n}_l\right) = \sum_{\substack{\vec{c}_n \text{ consistent} \\ \text{with } (\vec{d}_l, \vec{n}_l)}} \pi(\vec{c}_n)$$



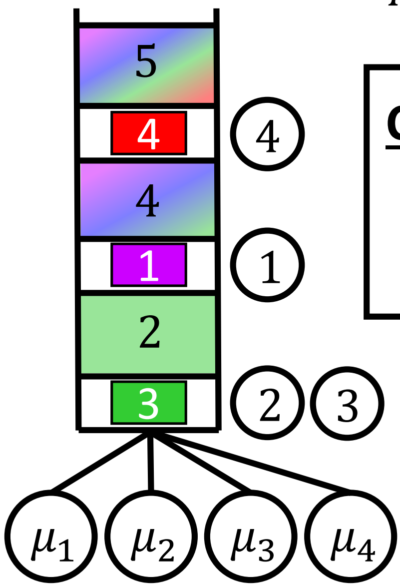
Partial Aggregation: Collaborative Model

Thm:

$$\pi\left(\vec{d}_l, \vec{n}_l\right) = \pi(0) \prod_{j=1}^{l} \frac{\lambda_{d_j}}{\mu\left(\vec{d}_j\right)} \left(\frac{\lambda\left(R\left(\vec{d}_j\right)\right)}{\mu\left(\vec{d}_j\right)} \right)^{n_j}$$

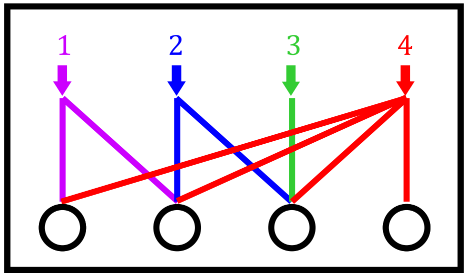
where $\pi(0)$ is the probability that the system is empty

N_j = # jobs between jth and $(j+1)$st jobs in service



Corollary:

$$N_j \left| \vec{d}_l =_{st} N_j \right| \vec{d}_j \sim \text{Geom}\left(1 - \frac{\lambda\left(R\left(\vec{d}_j\right)\right)}{\mu\left(\vec{d}_j\right)}\right)$$

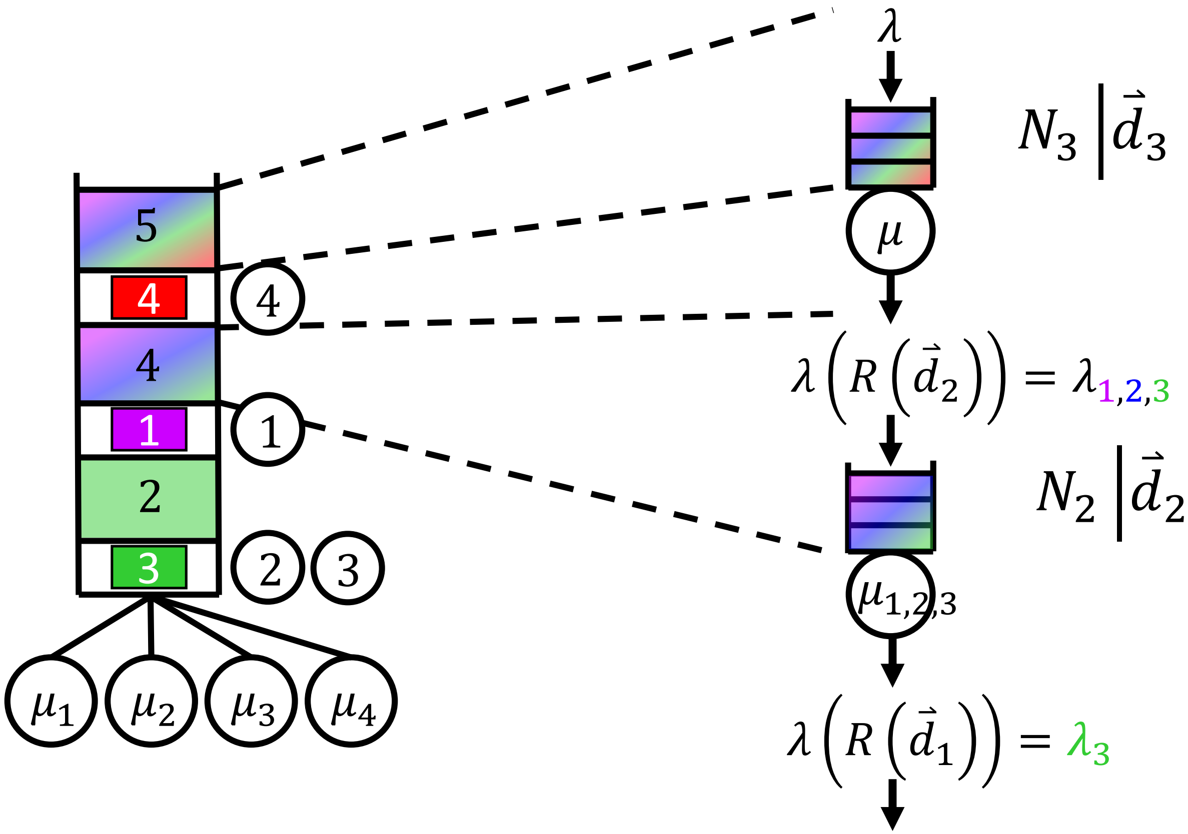


Partial Aggregation: Collaborative Model

Thm:

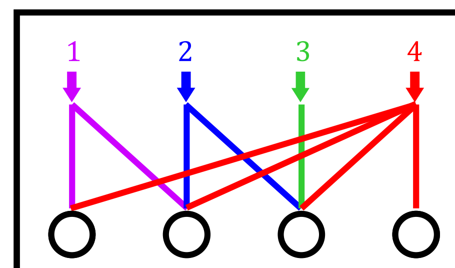
$$\pi\left(\vec{d}_l, \vec{n}_l\right) = \pi(0) \prod_{j=1}^{l} \frac{\lambda_{d_j}}{\mu\left(\vec{d}_j\right)} \left(\frac{\lambda\left(R\left(\vec{d}_j\right)\right)}{\mu\left(\vec{d}_j\right)} \right)^{n_j}$$

where $\pi(0)$ is the probability that the system is empty



$$N_3 \,\middle|\, \vec{d}_3 \sim \mathrm{Geom}\left(1 - \frac{\lambda}{\mu}\right)$$

$$N_2 \,\middle|\, \vec{d}_2 \sim \mathrm{Geom}\left(1 - \frac{\lambda_{1,2,3}}{\mu_{1,2,3}}\right)$$



Partial Aggregation: Collaborative Model

Thm:

$$\pi\left(\vec{d}_l, \vec{n}_l\right) = \pi(0) \prod_{j=1}^{l} \frac{\lambda_{d_j}}{\mu\left(\vec{d}_j\right)} \left(\frac{\lambda\left(R\left(\vec{d}_j\right)\right)}{\mu\left(\vec{d}_j\right)} \right)^{n_j}$$

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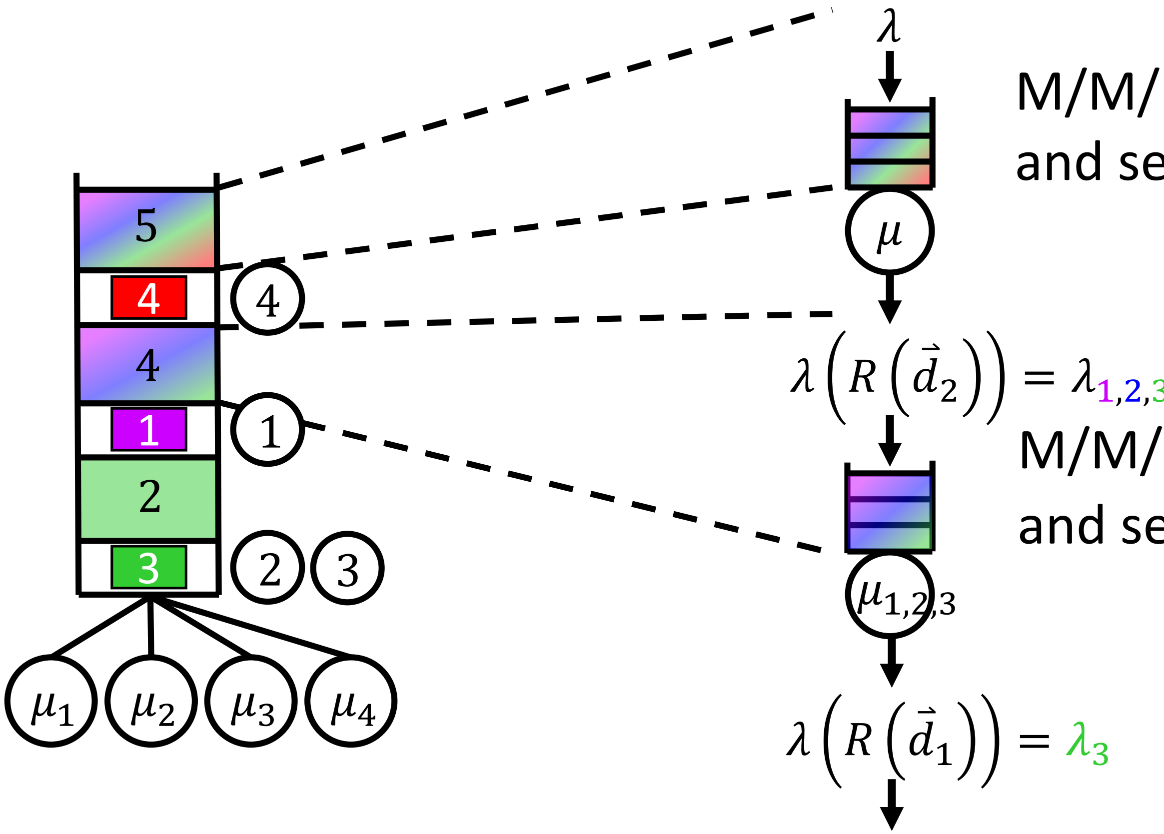
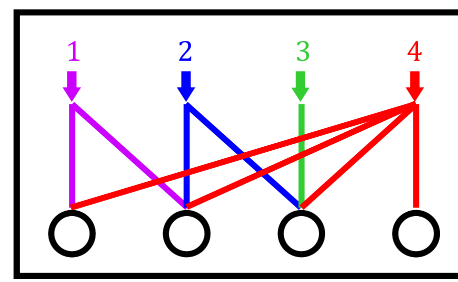


M/M/1 with arrival rate λ and service rate μ

M/M/1 with arrival rate $\lambda_{1,2,3}$ and service rate $\mu_{1,2,3,5}$

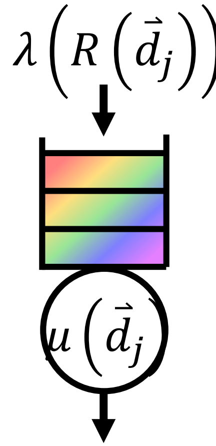


Performance Metrics: Collaborative Model

Corollary:

$$N_j \left| \vec{d}_l =_{st} N_j \right| \vec{d}_j \sim \text{Geom}\left(1 - \frac{\lambda\left(R\left(\vec{d}_j\right)\right)}{\mu\left(\vec{d}_j\right)}\right)$$

$N_j^{(i)}$ = # class-i jobs between jth and $(j+1)$st jobs in service



$$N_j \left| \vec{d}_j \sim \text{Geom}\left(1 - \frac{\lambda\left(R\left(\vec{d}_j\right)\right)}{\mu\left(\vec{d}_j\right)}\right)\right.$$

each is class 3

with probability $\frac{\lambda_3}{\lambda\left(R\left(\vec{d}_j\right)\right)}$

Lemma: If $N \sim \text{Geom}(1-p)$ and $N_j^{(i)} | N \sim \text{Binomial}\left(N, \frac{q_i}{p}\right)$, then

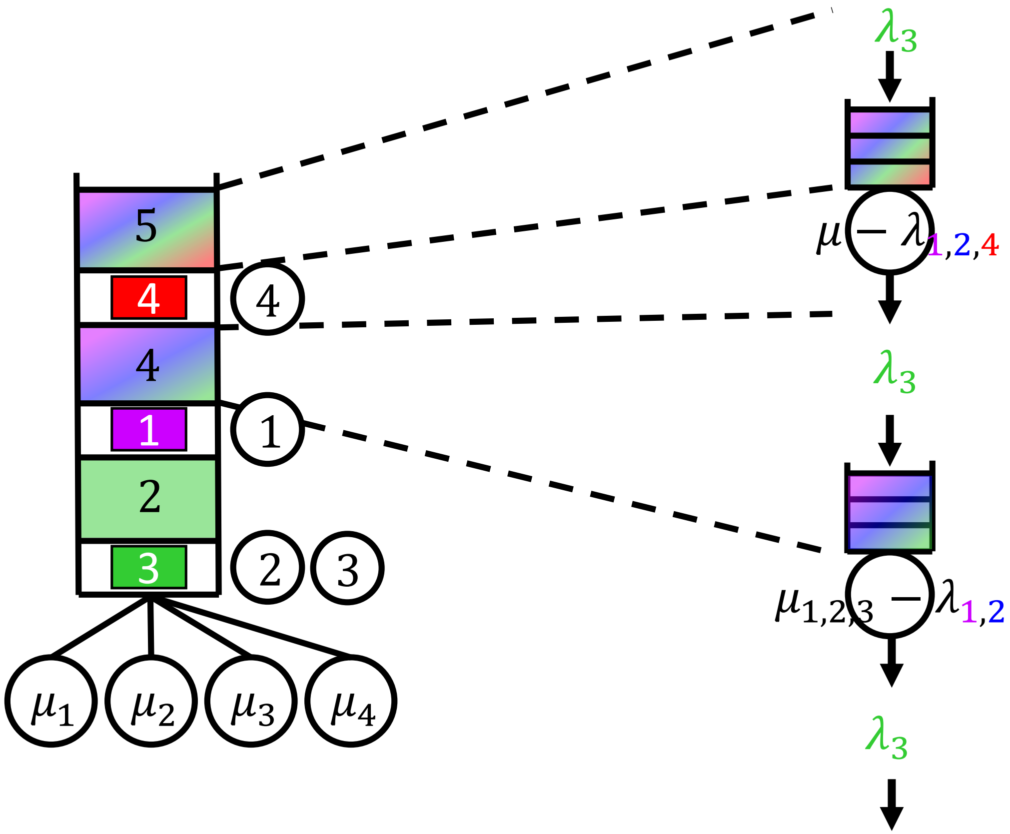
$$N_j^{(i)} \sim \text{Geom}\left(1 - \frac{q_i}{q_i + 1 - p}\right)$$

Let $p = \frac{\lambda\left(R\left(\vec{d}_j\right)\right)}{\mu\left(\vec{d}_j\right)}, \frac{q_i}{p} = \frac{\lambda_i}{\lambda\left(R\left(\vec{d}_j\right)\right)}$

Corollary: $N_j^{(i)} \left| \vec{d}_l =_{st} N_j^{(i)} \right| \vec{d}_j \sim \text{Geom}\left(1 - \frac{\lambda_i}{\mu\left(\vec{d}_j\right) - \lambda\left(R\left(\vec{d}_j\right)\right) + \lambda_i}\right)$

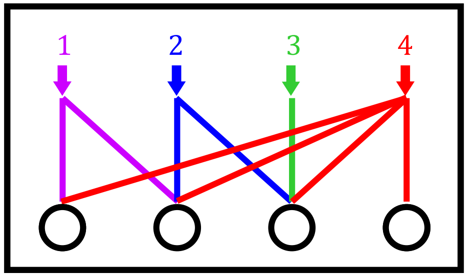
Performance Metrics: Collaborative Model

Corollary: $N_j^{(i)} \,\middle|\, \vec{d}_l =_{st} N_j^{(i)} \,\middle|\, \vec{d}_j \sim \text{Geom}\left(1 - \frac{\lambda_i}{\mu\left(\vec{d}_j\right) - \lambda\left(R\left(\vec{d}_j\right)\right) + \lambda_i}\right)$



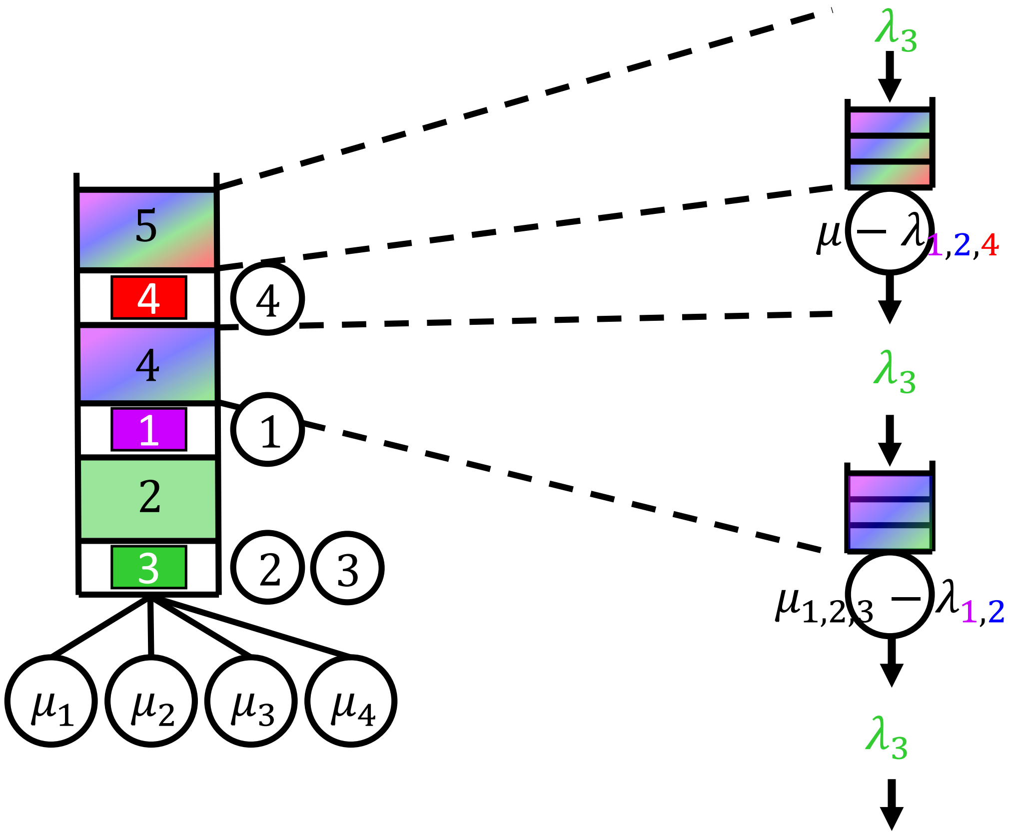
$$N_3^{(3)} \,\middle|\, \vec{d}_3 \sim \text{Geom}\left(1 - \frac{\lambda_3}{\mu - \lambda_{1,2,4}}\right)$$

$$N_2^{(3)} \,\middle|\, \vec{d}_2 \sim \text{Geom}\left(1 - \frac{\lambda_3}{\mu_{1,2,3} - \lambda_{1,2}}\right)$$



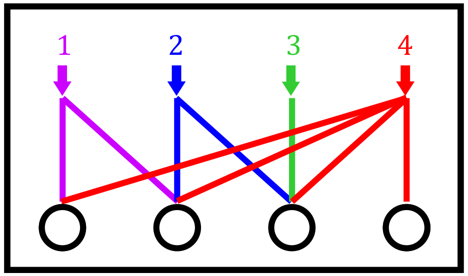
Performance Metrics: Collaborative Model

Corollary: $N_j^{(i)} \,\middle|\, \vec{d}_l =_{st} N_j^{(i)} \,\middle|\, \vec{d}_j \sim \text{Geom}\left(1 - \frac{\lambda_i}{\mu\left(\vec{d}_j\right) - \lambda\left(R\left(\vec{d}_j\right)\right) + \lambda_i}\right)$



Class 3: M/M/1 with arrival rate λ_3 and service rate $\mu - \lambda_{1,2,4}$

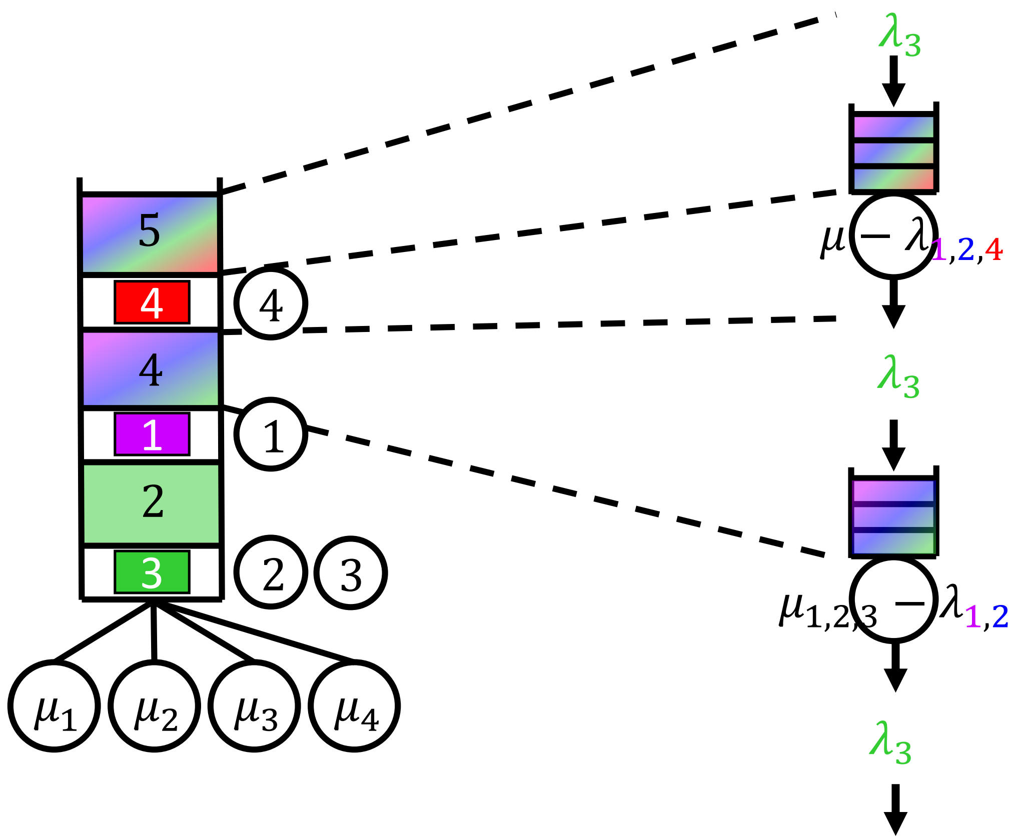
Class 3: M/M/1 with arrival rate λ_3 and service rate $\mu_{1,2,3} - \lambda_{1,2}$



Performance Metrics: Collaborative Model

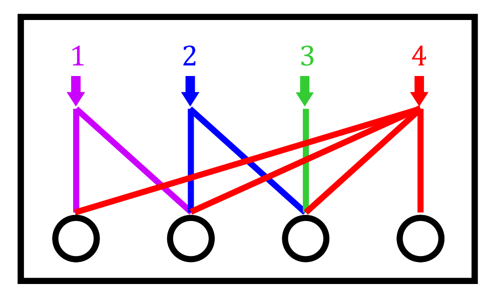
$W_Q^{(i)}\left(\vec{d}_l\right)$ = time in queue for a class-i job, given it sees service profile $\vec{d}_l$

$W_Q^{M/M/1}(\lambda, \mu)$ = time in queue in an M/M/1 with arrival rate λ and service rate μ



$W_Q^{M/M/1}(\lambda_3, \mu - \lambda_{1,2,4})$

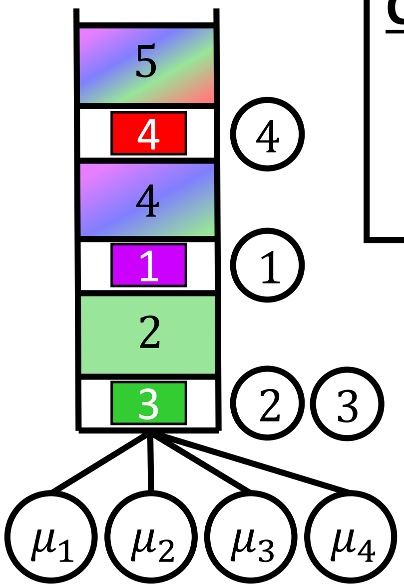
$W_Q^{M/M/1}(\lambda_3, \mu_{1,2,3} - \lambda_{1,2})$



Performance Metrics: Collaborative Model

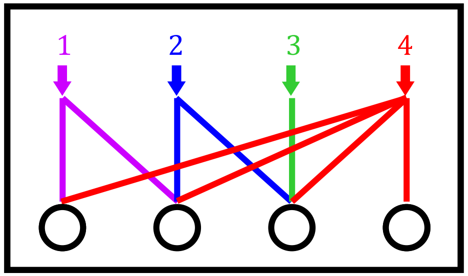
$W_Q^{(i)}\left(\vec{d}_l\right)$ = time in queue for a class-i job, given it sees service profile $\vec{d}_l$

$W_Q^{M/M/1}(\lambda,\mu)$ = time in queue in an M/M/1 with arrival rate λ and service rate μ



Corollary:

$$W_Q^{(i)}\left(\vec{d}_l\right) =_{st} \sum_{j:i\in R\left(\vec{d}_j\right)} W_Q^{M/M/1}\left(\lambda_i, \mu\left(\vec{d}_j\right) - \lambda\left(R\left(\vec{d}_j\right)\right) + \lambda_i\right)$$



Outline

- System description
- Review: reversibility
- Detailed states

  Collaborative model   Noncollaborative model

  Relating the models
- **Partial aggregation**

  Collaborative model   **Noncollaborative model**

  - Special case: fully flexible class
  - Special case: nested systems
- Related product-form systems

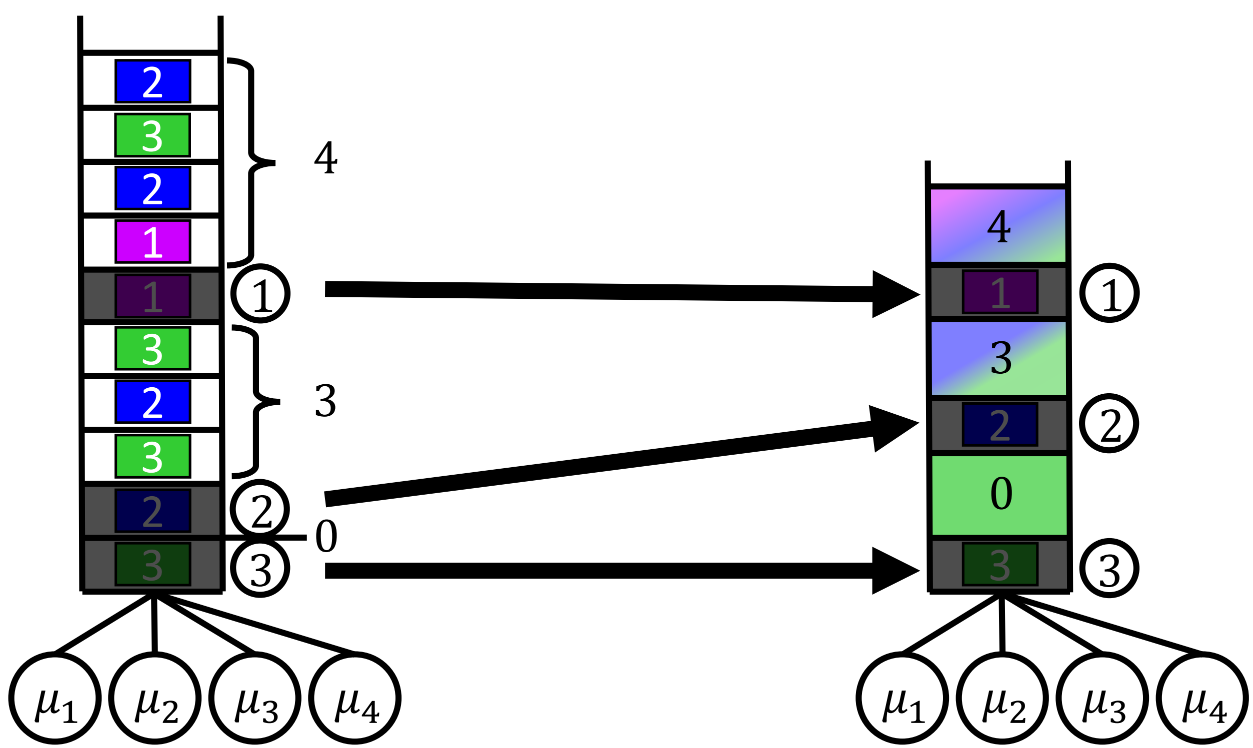
Partial Aggregation: Noncollaborative ALIS

$$\text{state}\left(\vec{n}_l; \vec{b}_l; \vec{s}_{M-l}\right) = \left(n_1, \dots, n_j, \dots, n_l; b_1, \dots, b_j, \dots, b_l; s_1, \dots, s_j, \dots, s_{M-l}\right)$$

number of jobs in the queue between the jth and $(j+1)$st jobs in service

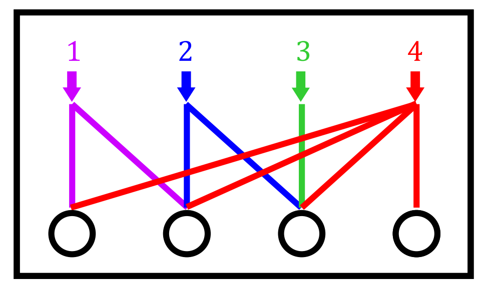
server that has been busy jth longest

server that has been idle jth longest



$\vec{c}_n = (3,3,3,1,2,3,2)$

$\left(\vec{n}_l; \vec{b}_l; \vec{s}_{M-l}\right) = (0,3,4; 3,2,1; 4)$



Partial Aggregation: Noncollaborative

Thm: Under ALIS,

$$\pi\left(\vec{n}_l; \vec{b}_l; \vec{s}_{M-l}\right) = \pi(0; \{b_1, \dots, b_M\}; 0) \prod_{j=1}^{l} \left(\frac{\lambda\left(R\left(\vec{b}_j\right)\right)}{\mu\left(\vec{b}_j\right)}\right)^{n_j} \prod_{j=1}^{l} \frac{1}{\mu\left(\vec{b}_j\right)} \prod_{j=1}^{M-l} \frac{1}{\lambda\left(\vec{s}_j\right)}$$

where $\vec{b}_l$ denotes the set of busy servers ordered by the arrival times of the jobs they are serving.

Thm: Under random assignment (with activation rate condition),

$$\pi\left(\vec{n}_l; \vec{b}_l\right) = \pi(0; 0) \prod_{j=1}^{l} \left(\frac{\lambda\left(R\left(\vec{b}_j\right)\right)}{\mu\left(\vec{b}_j\right)}\right)^{n_j} \prod_{j=1}^{l} \frac{\lambda^A_{b_j}\left(\vec{b}_{j-1}\right)}{\mu\left(\vec{b}_j\right)}$$

where $\vec{b}_l$ denotes the set of busy servers ordered by the arrival times of the jobs they are serving and $\lambda^A_{b_j}\left(\vec{b}_{j-1}\right)$ denotes the activation rate of server b_j, given $\vec{b}_{j-1}$.

[Adan, Visschers, Weiss]

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  - **Special case: fully flexible class**
  - Special case: nested systems
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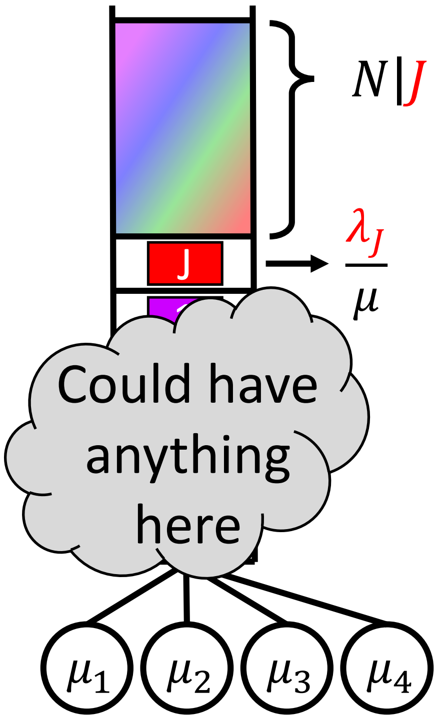
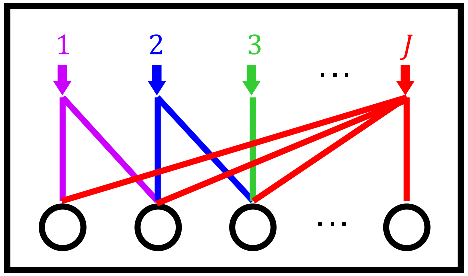
Fully Flexible Class: Collaborative Model



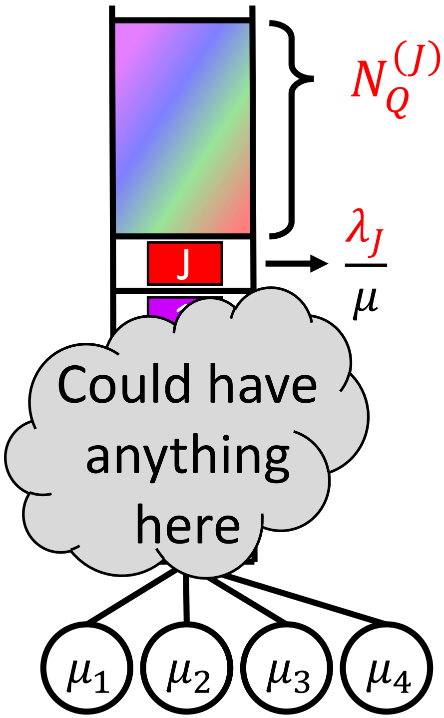
can aggregate over all classes

$$N|J \text{ in system} \sim \text{Geom}\left(1 - \frac{\lambda}{\mu}\right)$$

$\frac{\lambda_J}{\mu}$



Fully Flexible Class: Collaborative Model



$N_Q^{(J)} | J \text{ in system} \sim \text{Geom}\left(1 - \frac{\lambda_J}{\mu - \lambda + \lambda_J}\right)$ (by the same lemma used for partial aggregation)

$$N^{(J)} = \Pr\{J \text{ in system}\} \cdot \left(1 + N_Q^{(J)} | J \text{ in system}\right)$$

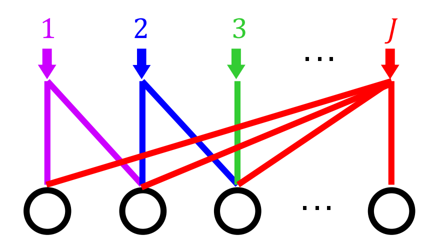
$$\sim \text{Geom}\left(1 - \frac{\lambda_J}{\mu - \lambda + \lambda_J}\right)$$

as long as $\Pr\{J \text{ in system}\} = \rho_J = \frac{\lambda_J}{\mu - \lambda + \lambda_J}$

(short proof, won't show here)

Fully flexible class experiences M/M/1 response time:

$$W^J =_{st} W^{M/M/1}(\lambda_J, \mu - \lambda + \lambda_J) =_{st} W^{M/M/1}(\lambda, \mu)$$



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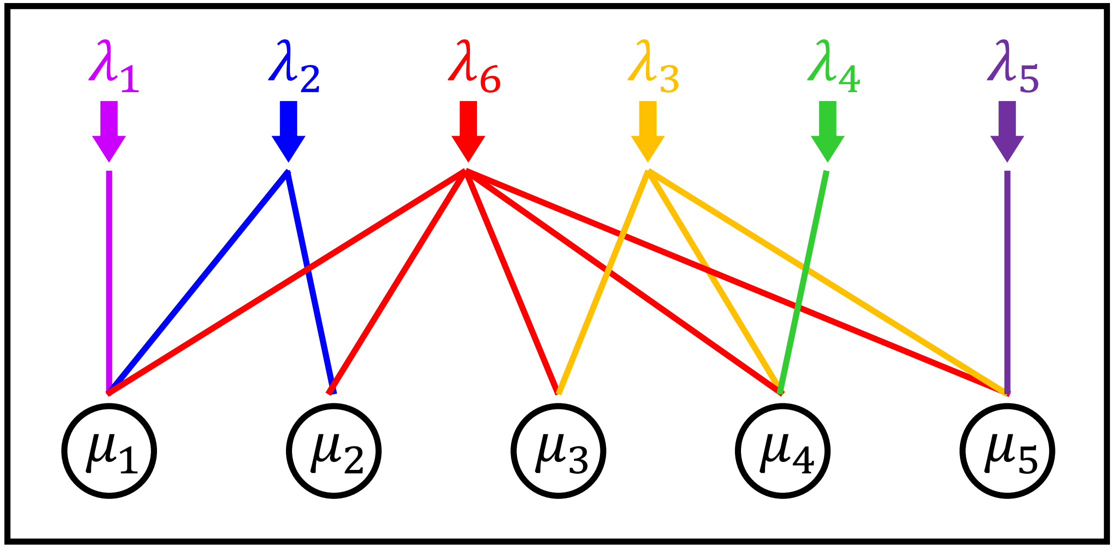
Nested Systems: Collaborative Model

For every pair of job classes i and j, either:

- $S_i \subset S_j$
- $S_j \subset S_i$
- $S_i \cap S_j = \emptyset$

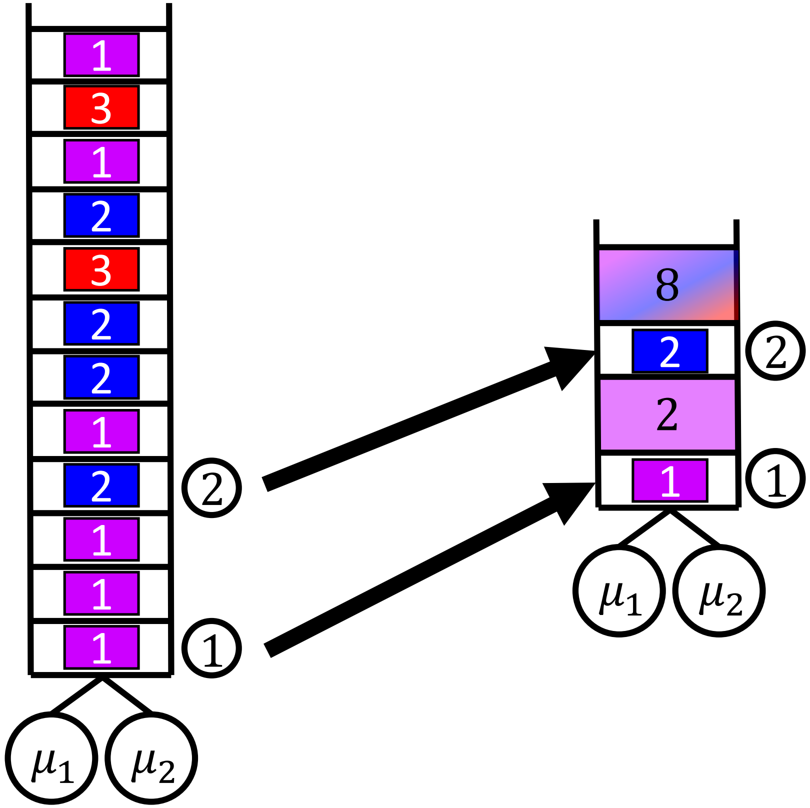


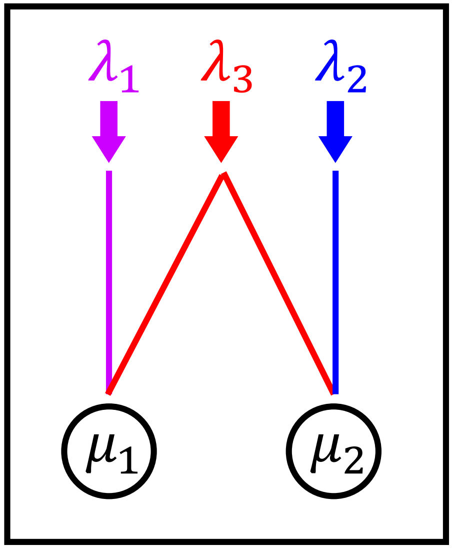
Recursive structure: removing the fully flexible class leaves two smaller nested systems

W System: Collaborative Model

Detailed states

Partially Aggregated States



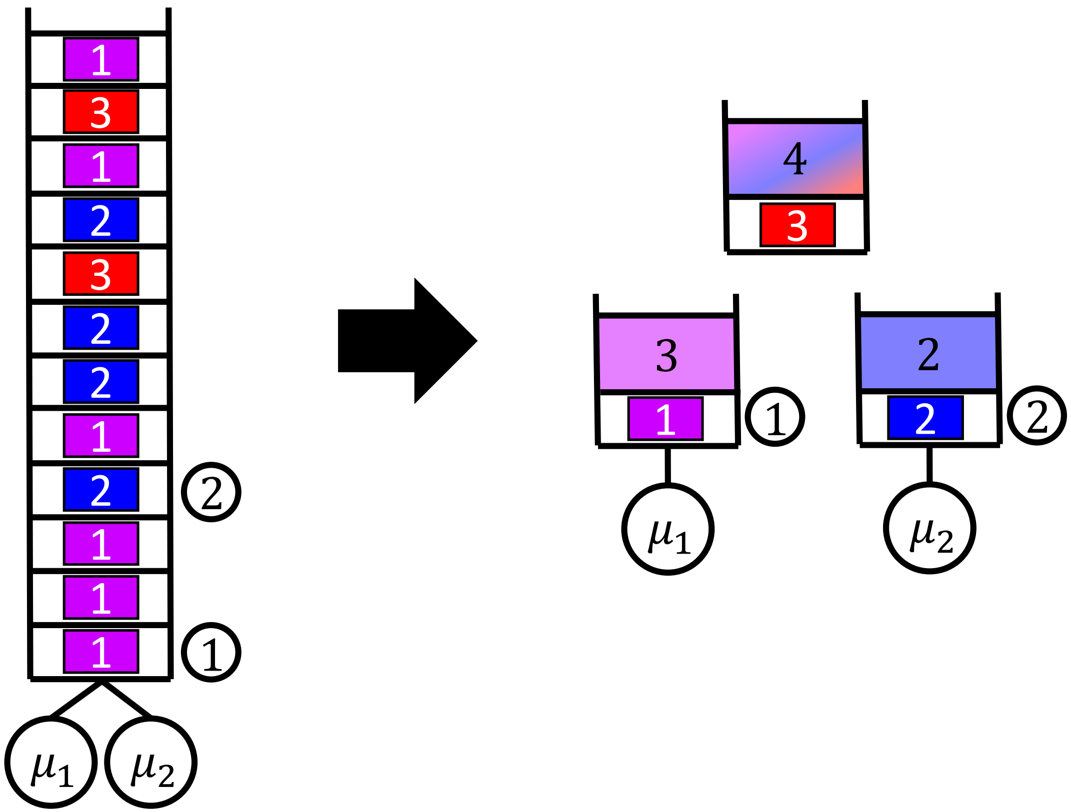


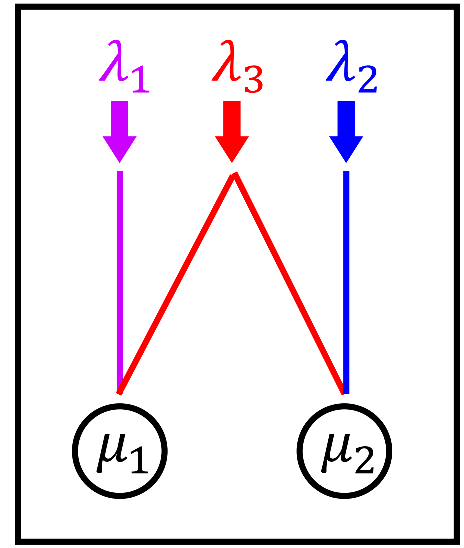
Observation: a class 1 job is never blocked from service by a class 2 job

W System: Collaborative Model

Detailed states

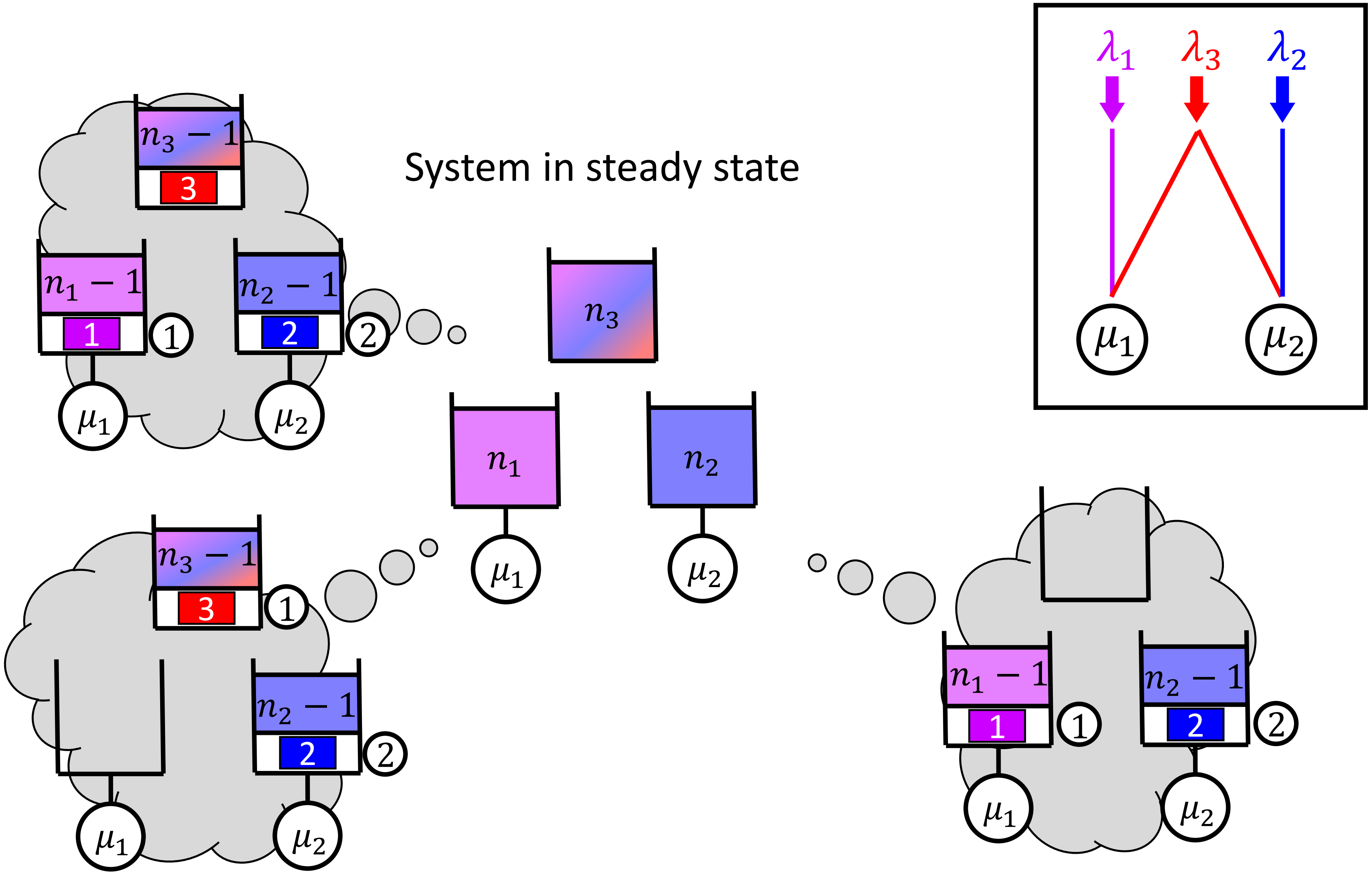
Partially Aggregated States for Nested Systems





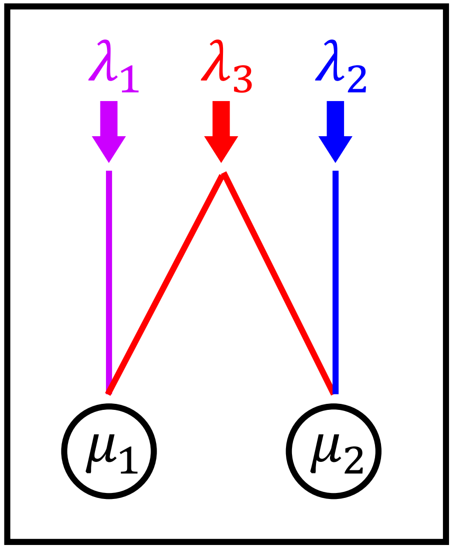
Observation: a class 1 job is never blocked from service by a class 2 job

W System: Collaborative Model



W System: Collaborative Model

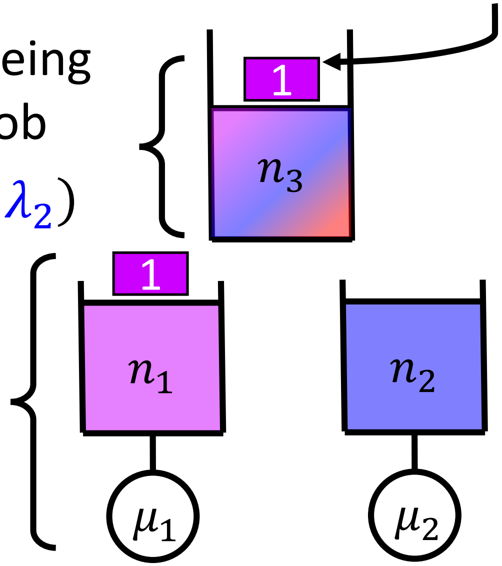
Understanding **class 1** response time



tagged **class 1** arrival

experiences queueing time of a **class 3** job

$W^Q(\lambda_3, \mu - \lambda_1 - \lambda_2)$



experiences response time of a **class 1** job

$W(\lambda_1, \mu_1)$

$$W_1 \sim W^Q(\lambda_3, \mu - \lambda_1 - \lambda_2) + W(\lambda_1, \mu_1)$$

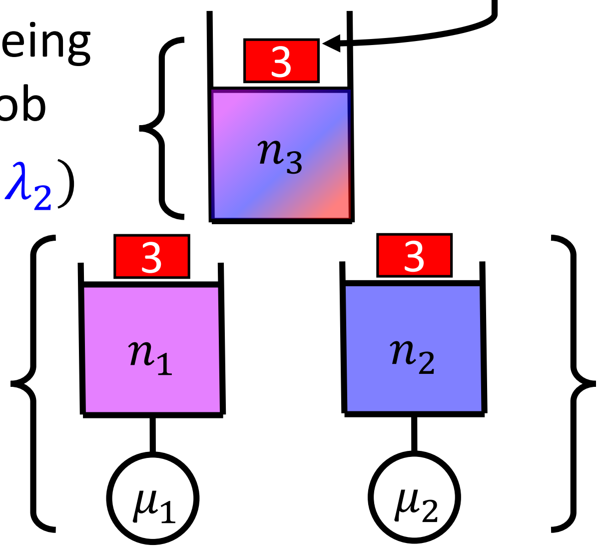
W System: Collaborative Model

Understanding **class 3** response time

tagged **class 3** arrival

experiences queueing time of a **class 3** job

$W^Q(\lambda_3, \mu - \lambda_1 - \lambda_2)$

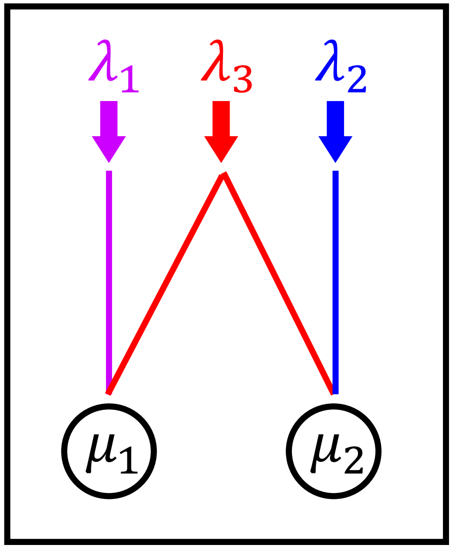


experiences response time of a **class 1** job

$W(\lambda_1, \mu_1)$

experiences response time of a **class 2** job

$W(\lambda_2, \mu_2)$



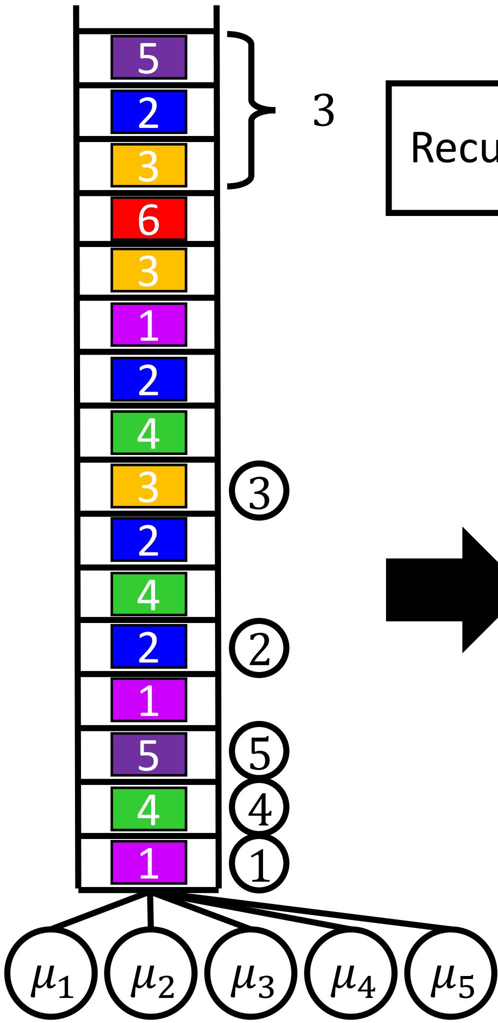
$$W_3 \sim W^Q(\lambda_3, \mu - \lambda_1 - \lambda_2) + \min\{W(\lambda_1, \mu_1), W(\lambda_2, \mu_2)\}$$

$$\sim W^Q(\lambda_3, \mu - \lambda_1 - \lambda_2) + \min\{\mathrm{Exp}(\mu_1 - \lambda_1), \mathrm{Exp}\{\mu_2 - \lambda_2\}\}$$

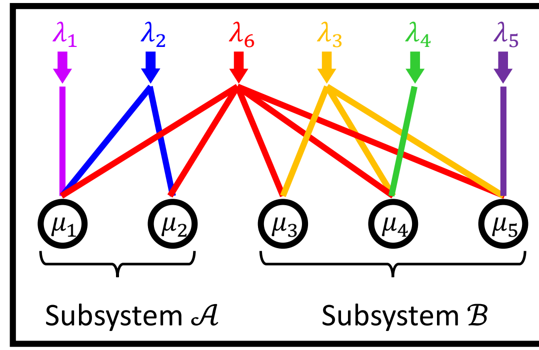
$$\sim W^Q(\lambda_3, \mu - \lambda_1 - \lambda_2) + \mathrm{Exp}(\mu - \lambda_1 - \lambda_2)$$

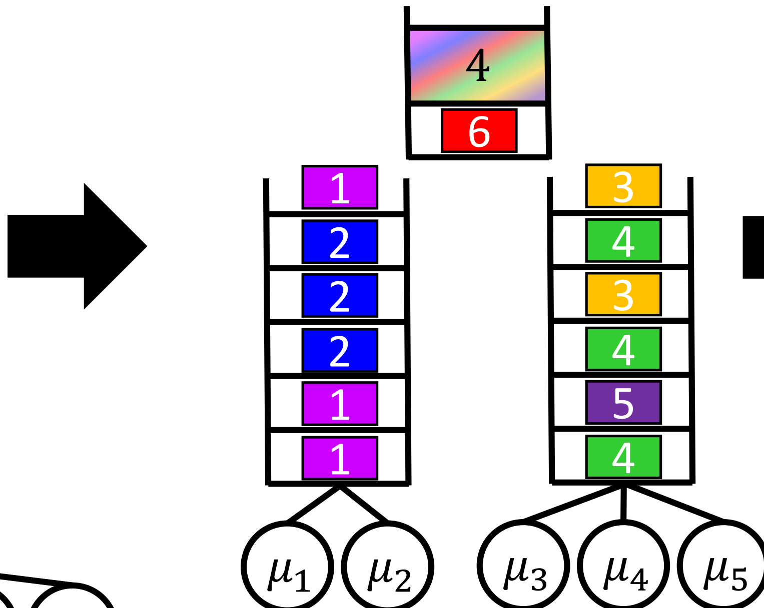
$$\sim W(\lambda_3, \mu - \lambda_1 - \lambda_2)$$

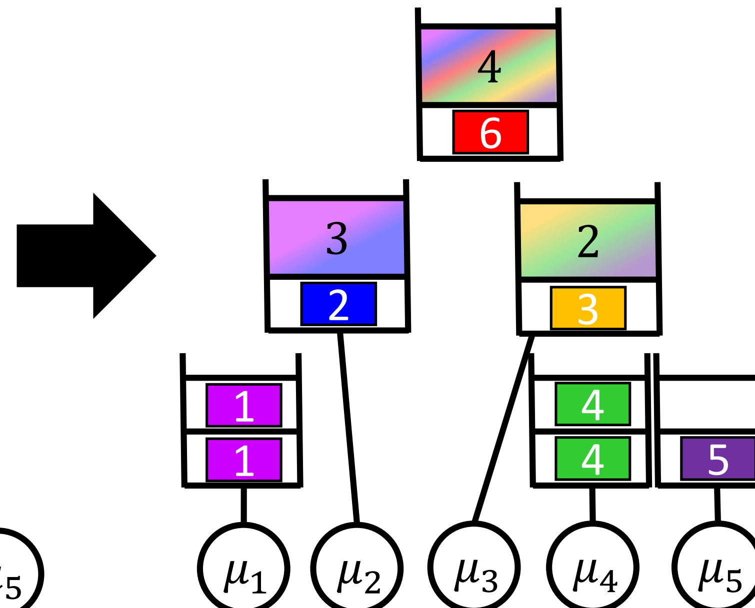
Nested Systems: Collaborative Model



Recursively defined state: $\left(n_J, J, \begin{matrix}(\mathcal{A})\\(\mathcal{B})\end{matrix}\right)$







$$\left(4, 6, \begin{matrix}\left(3, 2, \begin{matrix}1,1\\0\end{matrix}\right)\\ \left(2, 3, \begin{matrix}1,4\\0,5\end{matrix}\right)\end{matrix}\right)$$

$$\vec{c}_{16} = (1,4,5,1,2,4,2,3,4,2,1,3,6,3,2,5)$$

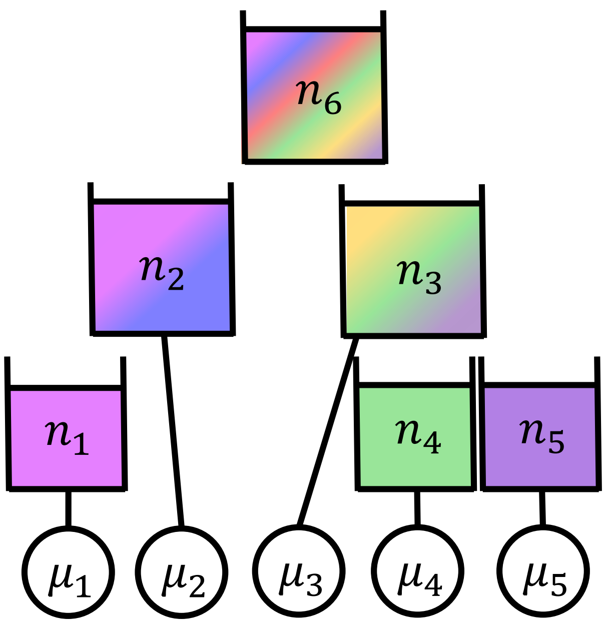
Nested Systems: Collaborative Model

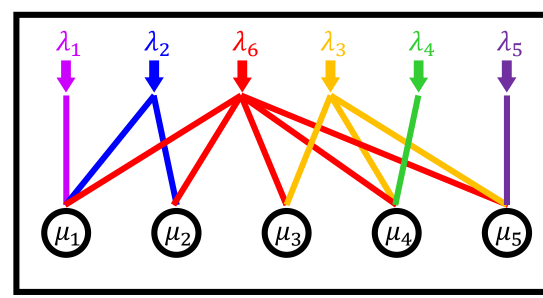
Thm:

$$\pi\left(n_J, J, \begin{matrix}(\mathcal{A})\\(\mathcal{B})\end{matrix}\right) = (1-\rho_J)\left(\frac{\lambda\left(R(S_J)\right)}{\mu(S_J)}\right)^{n_J}\left(\frac{\lambda_J}{\mu(S_J)}\right)\pi(\mathcal{A})\pi(\mathcal{B})$$

where $\rho_i = \frac{\lambda_i}{\mu(S_i) - \lambda\left(R(S_i)\right) + \lambda_i}$

[Gardner, Harchol-Balter, Hyytiä, Righter]



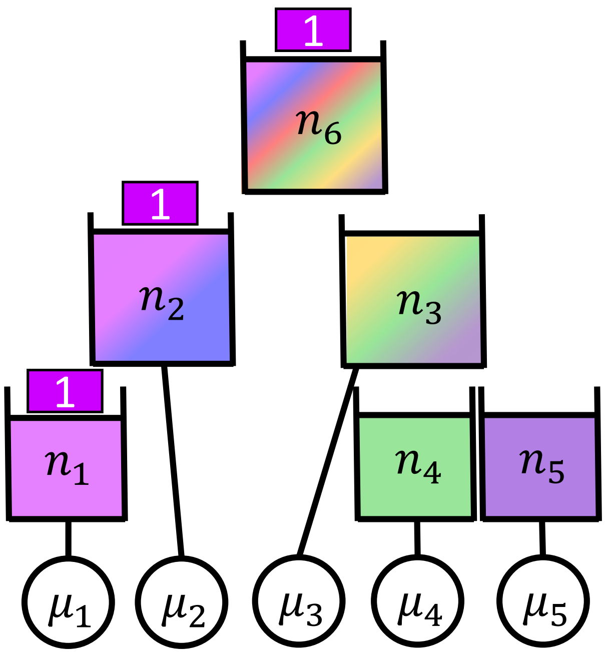


Nested Systems: Collaborative Model

Deriving Performance Metrics

Corollary:

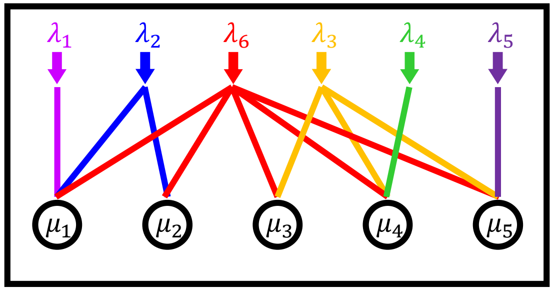
$$W_i =_{st} W\left(\lambda\left(R(S_i)\right), \mu(S_i)\right) + \sum_{j:S_i \subset S_j} W^Q\left(\lambda_j, \mu(S_j) - \lambda\left(R(S_j)\right) + \lambda_j\right)$$



ex. $W_1 =_{st} \quad W^Q(\lambda, \mu - \lambda + \lambda_6)$

$+W^Q(\lambda_2, \mu_{1,2} - \lambda_{1,2} + \lambda_2)$

$+W(\lambda_1, \mu_1)$



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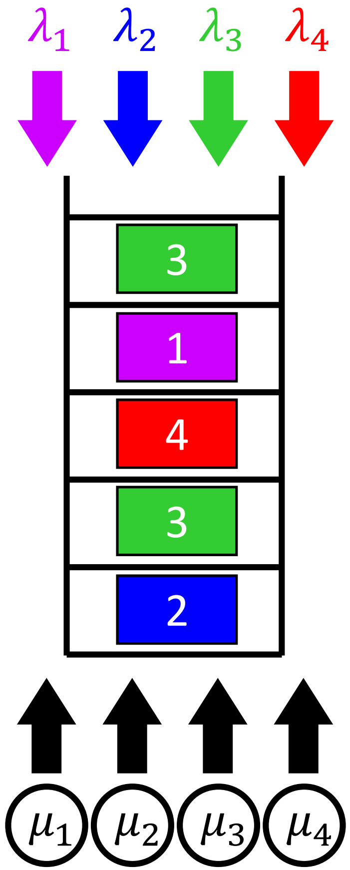
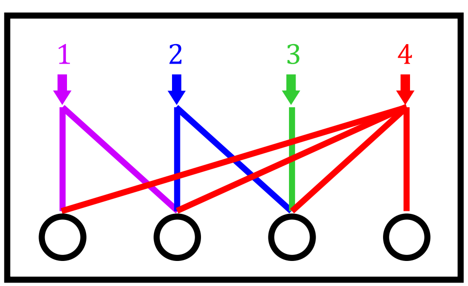
- **Related product-form systems**

Matching Models





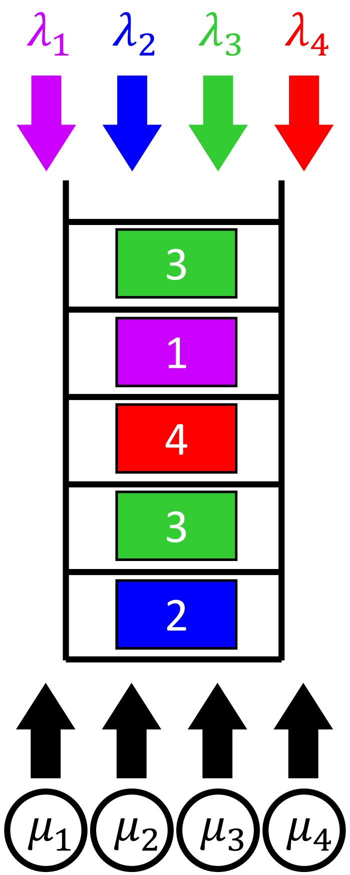
Model 1: arriving jobs wait in the queue for a compatible server, arriving servers match the first compatible job and leave the system (even if unmatched)

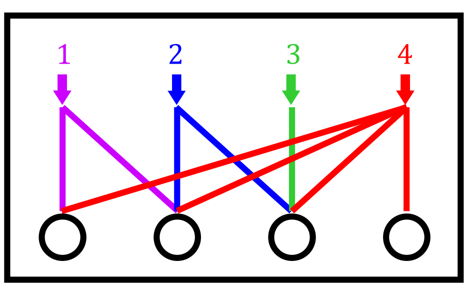
Sample path equivalences for $\vec{c}_n$:

Noncollaborative model given busy (**queue** state) ⟷ Model 1 ⟷ Collaborative model (**system** state)

[Adan, Kleiner, Righter, Weiss]

Matching Models





Other matching models with product form

Model 2 (FCFS infinite bipartite matching): arrivals are job-server pairs, both of which can queue; arriving jobs (servers) match the first compatible server (job) if any, otherwise join queue

[Adan, Busic, Gupta, Mairesse, Weiss]

Thm:

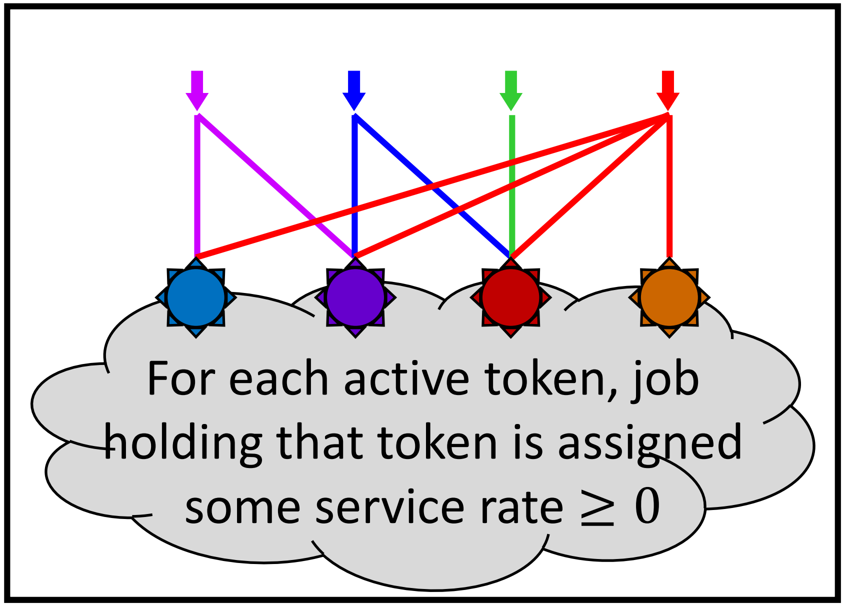
$$\pi(\vec{c}_n;\vec{s}_n) = \pi(0;0)\prod_{i=1}^{n}\frac{\lambda_{c_i}}{\mu(\vec{c}_i)}\frac{\mu_{s_i}}{\lambda(\vec{s}_i)}$$

Model 3: individual arrivals, non-bipartite matching graph, arrivals match the first compatible item in queue if any, otherwise join queue

[Busic, Mairesse, Moyal, Perry]

Unifying Token Model

- Set of tokens, bipartite matching between jobs and tokens
- Arriving job takes compatible token if one is available (assignment condition)
- Only jobs with tokens can be served
- OI condition on tokens defines service process



state $(\vec{c}_n; \vec{a}_m)$

active tokens in the order in which they became active

Thm: $$\pi(\vec{c}_n; \vec{a}_m) = B_3 \prod_{i=1}^{n} \frac{\lambda_{c_i}}{\mu(\vec{c}_i)} \prod_{l=1}^{m} \frac{\lambda^A_{a_l}(\vec{a}_{l-1})}{\mu(\vec{a}_l)}$$

[Ayesta, Bodas, Dorsman, Verloop]

Unifying Token Model

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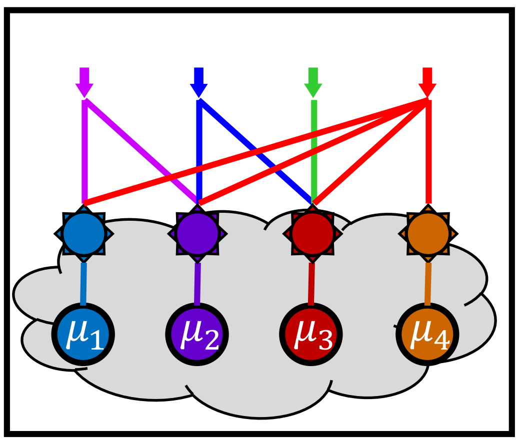
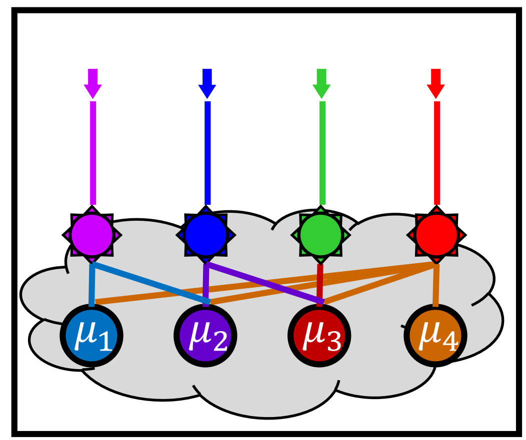
Noncollaborative Model



tokens ↔ servers

Collaborative Model



tokens ↔ job classes

…but much more general than these two models

e.g., MSCCC queue [Ayesta, Bodas, Dorsman, Verloop]

Summary and Conclusions: Part 1

Detailed States

Collaborative (C) model

$\vec{c}_n$

classes of all jobs in the **system**

Noncollaborative (NC) model

ALIS: $(\vec{c}_n; \vec{s}_k)$

RAND: $(\vec{c}_n; b_m)$

classes of all jobs in the **queue**

servers in order they became **idle**

servers in order they became **busy**

Conditioned on all servers being busy in NC, the noncollaborative **queue** has the same stationary distribution as the collaborative **system**

Partially Aggregated States

Collaborative (C) model

$(\vec{d}_l, \vec{n}_l)$

classes of all jobs in **service**

number of jobs in queue between jobs in service

Noncollaborative (NC) model

ALIS: $(\vec{n}_l; \vec{b}_l; \vec{s}_{M-l})$

RAND: $(\vec{n}_l; \vec{b}_l)$

Fully flexible class experiences M/M/1 response time

Easy to derive per-class response time distributions in nested systems (C)