Practice with Induction

1. Prove that for all $n > 0$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Solution: Base case: n = 1.
When $n = 1$, we have

$$\sum_{i=1}^{n} i = \sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}.$$

Inductive hypothesis: For some $n \geq 1$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Inductive step: Assume the inductive hypothesis holds for n. We will show that $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$.
We have:

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\ &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

as desired. Here the second line follows from the inductive hypothesis. Hence we have that for all $n > 0$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. $\square$

2. Prove that for all $n > 0$, $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$ if $|x| < 1$. This is called the finite geometric series.

Solution: Base case: $n = 1$.
When $n = 1$, we have

$$\sum_{i=0}^{n} x^i = \sum_{i=0}^{1} x^i = x^0 + x^1 = 1 + x = \frac{(1+x)(1-x)}{1-x} = \frac{1-x^2}{1-x}$$

as desired.

Inductive hypothesis: For some $n \geq 1$, $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$.

Inductive step: Assume the inductive hypothesis holds for n. We will show that $\sum_{i=0}^{n+1} x^i = \frac{1-x^{n+2}}{1-x}$.

We have:

$$\begin{aligned}\sum_{i=0}^{n+1} x^i &= \sum_{i=0}^{n} x^i + x^{n+1} \\ &= \frac{1-x^{n+1}}{1-x} + x^{n+1} \\ &= \frac{1-x^{n+1}+(1-x)x^{n+1}}{1-x} \\ &= \frac{1-x^{n+1}+x^{n+1}-x^{n+2}}{1-x} \\ &= \frac{1-x^{n+2}}{1-x}\end{aligned}$$

as desired. Here the second line follows from the inductive hypothesis. Hence we have that for all $n > 0$, $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$. $\square$

Note that if we take the limit of this series as $n \to \infty$, we get $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$. This is called the infinite geometric series, and is another useful summation identity.

3. Let $T(n) = \begin{cases} T(n/2) + c_1 n & n > 1 \\ c_2 & n = 1 \end{cases}$. Prove that $T(n) \in O(n)$.

Solution: Let $c = 2c_1 + c_2$ and $n_0 = 2$. We will show that for all $n \geq n_0$, $T(n) \leq cn$.
Base case: n=2.
When $n = 2$, we have

$$T(2) = T(2/2) + 2c_1 = T(1) + 2c_1 = c_2 + 2c_1 = c \leq 2c.$$

Inductive hypothesis: For some $n \geq 2$, $T(n) \leq cn$.

Inductive step: Assume the inductive hypothesis holds for n. We will show that $T(2n) \leq 2cn$. We have:

$$\begin{aligned}T(2n) &= T(2n/2) + 2c_1 n \\ &= T(n) + 2c_1 n \\ &\leq cn + 2c_1 n \\ &= (2c_1 + c)n \\ &\leq (4c_1 + c_2)n \\ &\leq (4c_1 + 2c_2)n \\ &= 2cn\end{aligned}$$

as desired. Here the third line follows from the inductive hypothesis and the fifth line follows from plugging in $c = 2c_1 + c_2$. Hence we have shown that there exists a c (namely $c = 2c_1 + c_2$) and

an n_0 (namely $n_0 = 2$) such that for all $n \geq n_0$, $T(n) \leq 2cn$. Hence by definition, $T(n) \in O(n)$.
$\square$

4. Let $T(n) = \begin{cases} 2T(n-1) + n & n > 1 \\ 1 & n = 1 \end{cases}$. Prove that $T(n) = 2^{n+1} - n - 2$.

Solution: Base case: $n = 1$.
When $n = 1$, we have

$$T(1) = 1 = 2^2 - 1 - 2.$$

Inductive hypothesis: For some $n \geq 1$, $T(n) = 2^{n+1} - n - 2$.

Inductive step: Assume the inductive hypothesis holds for n. We will show that $T(n+1) = 2^{n+2} - (n+1) - 2$. We have:

$$\begin{aligned} T(n+1) &= 2T(n) + (n+1) \\ &= 2(2^{n+1} - n - 2) + (n+1) \\ &= 2^{n+2} - 2n - 4 + n + 1 \\ &= 2^{n+2} - n - 3 \\ &= 2^{n+2} - (n+1) - 2 \end{aligned}$$

as desired. Hence we have shown that for all $n \geq 1$, $T(n) = 2^{n+1} - n - 2$. $\square$