Network Flow

The Network Flow Problem

Input:

- A directed, weighted graph $G = (V, E)$
- For each edge $e = (u, v) \in E$, a positive integer capacity $c(e)$
- A source node $s \in V$, where s has no incoming edges
- A target node $t \in V$, where t has no outgoing edges

Output: A valid flow f of maximum value (that is, $\arg\max_f \{v(f)\}$).

Notation

- A *flow* f is a function that maps each edge $e = (u, v)$ in a graph $G = (V, E)$ to a nonnegative number $f(e) = f(u, v)$ representing the amount of flow sent along that edge.
- The *value* of a flow, denoted $v(f)$, is the total amount of flow sent from the source node s to the target node t, where $v(f) = \sum_{(s,u)} f(s, u)$.
- Any valid flow must satisfy the following two conditions:
  - **Capacity condition:** For each edge $e \in E$, the amount of flow sent along the edge must be at most the edge's capacity. That is, $0 \leq f(e) \leq c(e)$ for all edges e, where $c(e)$ is the capacity (weight) of edge e.
  - **Conservation condition:** For each node $v \in V$ ($v \neq s, t$), the amount of flow into the node must be equal to the amount of flow out of the node. That is, $\sum_{u,v} f(u, v) = \sum_{v,w} f(v, w)$ for all nodes v.