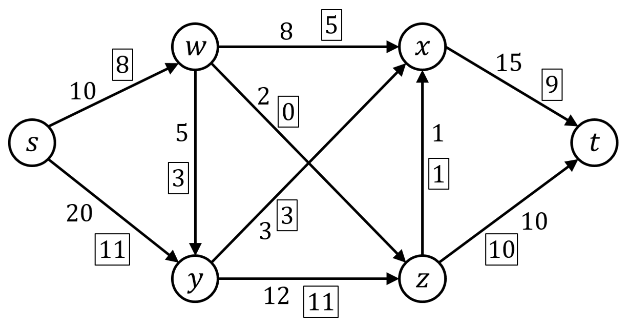
Part I: s-t Cuts

An s-t *cut* in a graph G is a partition of the vertices in G into sets (A, B) such that $s \in A$ and $t \in B$. The *capacity* of an s-t cut (A, B), denoted $c(A, B)$, is the sum of the capacities of all edges out of A:

$$c(A, B) = \sum_{e \text{ out of } A} c(e)$$

Here is a graph (numbers in boxes represent the flow along an edge, "unadorned" numbers represent the edge capacity):



1) Let $A = \{s, w, y\}$, $B = \{x, z, t\}$. What is the capacity of the cut?

We have defined the value of a flow f to be the total amount of flow leaving s:

$$v(f) = f^{\text{out}}(s) = \sum_{(s,u)} f(s, u)$$

2) In the above graph, what is the value of the flow leaving s?

The flow across a cut (A, B) is defined to be the net amount of flow leaving A; that is, the total amount of flow leaving A minus the total amount of flow entering A, or $f^{\text{out}}(A) - f^{\text{in}}(A)$.

3) Let $A = \{s, w, y\}$, $B = \{x, z, t\}$. What is the value of the flow across this cut?

4) Let $A = \{s, y, x\}$, $B = \{w, z, t\}$. What is the value of the flow across this cut?

5) What can you say, in general, about the value of a flow f with respect to the flow across any cut (A, B)? Why is this true?

(★)

(write an expression in this box that says something about $v(f)$)

Make sure everyone in your group understands what is going on. At this point, ask me for the next page.

Part II: An Idea for a Cut

Claim 1. *If f is any valid flow, and (A, B) is any s-t cut, then $v(f) \leq c(A, B)$.*

6) Prove this claim in five lines, starting with the expression you wrote in (★).

We have now shown that *every* s-t cut gives an upper bound on the value of any valid flow. If we can show that Ford-Fulkerson returns a flow $\hat{f}$ with value $v(\hat{f}) = c(A^*, B^*)$ for some cut (A^*, B^*), then we will know that $\hat{f}$ is a max flow.

Suppose we have a graph G and a flow f for which there is no s-t path in the residual graph G_f. Consider the following idea for a cut: Let A^* be the set of nodes v that are reachable from s in the residual graph, and let $B^* = V - A^*$).

7) How do we know that (A^*, B^*) is in fact an s-t cut?

8) Suppose edge $e = (u, v)$ has $u \in A^*$ and $v \in B^*$. What *must* be the value of $f(e)$? Why?

9) Suppose edge $e = (u, v)$ has $u \in B^*$ and $v \in A^*$. What *must* be the value of $f(e)$? Why?

10) Now use (★) to show in four lines that $v(f) = c(A^*, B^*)$.

At this point, ask me for the next page.

Part III: Proving Ford-Fulkerson Finds a Max Flow

It's time to put it all together and prove that Ford-Fulkerson successfully computes a max flow.

11) When does Ford-Fulkerson terminate? In answering this, review what Ford-Fulkerson does, and make sure everyone in your group understands what the algorithm is doing.

12) Consider the cut (A^*, B^*) as defined in Part II. What must be true about the value of the flow produced by Ford-Fulkerson, $\hat{f}$, with respect to (A^*, B^*)?

13) So why is $\hat{f}$ a max flow?

What we have actually shown here is the following key theorem:

Theorem 1. ***[Max-Flow Min-Cut]*** *In every flow network, the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut.*

This is important because when we solve applied problems, sometimes we are looking for a max flow and sometimes we are looking for a min cut. The Max-Flow Min-Cut Theorem tells us that these always have the same value.

14) Suppose we have computed a max s-t flow f using Ford-Fulkerson. How can we compute a min s-t cut in time $O(m)$, where m is the number of edges in our graph?