SOLUTIONS

Practice with Asymptotic Notation

****Turn in at the start of class on Monday. This will not be graded, and is simply for your benefit and to let me know whether the class would benefit from further review.****

**Def 1.** \( O(g(n)) = \{ f(n) : \exists c, n_0 > 0 \text{ such that } 0 \leq f(n) \leq c \cdot g(n) \forall n \geq n_0 \} \)

**Def 2.** \( \Omega(g(n)) = \{ f(n) : \exists c, n_0 > 0 \text{ such that } 0 \leq c \cdot g(n) \leq f(n) \forall n \geq n_0 \} \)

**Def 3.** \( \Theta(g(n)) = \{ f(n) : \exists c_1, c_2, n_0 > 0 \text{ such that } 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \forall n \geq n_0 \} \)

**Def 4.** \( o(g(n)) = \{ f(n) : \forall c > 0, \exists n_0 > 0 \text{ such that } 0 \leq f(n) < c \cdot g(n) \forall n \geq n_0 \} \)

**Def 5.** \( \omega(g(n)) = \{ f(n) : \forall c > 0, \exists n_0 > 0 \text{ such that } 0 \leq c \cdot g(n) < f(n) \forall n \geq n_0 \} \)

Recall that there is an easy test that can be used to compare two functions \( f(n) \) and \( g(n) \): Compute

\[
C = \lim_{n \to \infty} \frac{f(n)}{g(n)}.
\]

1. Fill in the table:

<table>
<thead>
<tr>
<th>If...</th>
<th>Then...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C = 0 )</td>
<td>( f(n) \in o(g(n)) ) and ( g(n) \in \omega(f(n)) )</td>
</tr>
<tr>
<td>( 0 &lt; C &lt; \infty )</td>
<td>( f(n) \in \Theta(g(n)) ) and ( g(n) \in \Theta(f(n)) )</td>
</tr>
<tr>
<td>( C = \infty )</td>
<td>( f(n) \in \omega(g(n)) ) and ( g(n) \in o(f(n)) )</td>
</tr>
</tbody>
</table>

2. Prove that \( f(n) = 3n + 16 \in \Theta(n) \).

**Solution:** We will show that \( \exists c_1, c_2, n_0 > 0 \text{ such that } \forall n \geq n_0, 0 \leq c_1 n \leq f(n) \leq c_2 n \). Let \( c_1 = 4, c_2 = 3, \) and \( n_0 = 16 \). Then \( \forall n \geq n_0 = 16, \) we have \( f(n) = 3n + 16 \leq 3n + n = 4n \). Similarly, \( \forall n \geq n_0 = 16, \) we have \( f(n) = 3n + 16 \geq 3n \). Hence by definition, \( f(n) \in \Theta(n) \). \( \square \)

**Alternative Solution:** We will use the limit rule:

\[
\lim_{n \to \infty} \frac{3n + 16}{n} = \lim_{n \to \infty} 3 + \frac{16}{n} = 3.
\]

Since \( 0 < 3 < \infty \), we fall into case (2) of the above chart, so \( f(n) = 3n + 16 \in \Theta(n) \). \( \square \)
3. Prove that $\Theta(\log_a n) = \Theta(\log_b n)$ for all constants $a, b > 0$.

**Solution:** Let $f(n)$ be an arbitrary function such that $f(n) \in \Theta(\log_a n)$. Then there exist constants $c_1, c_2, n_0 > 0$ such that $\forall n \geq n_0,$

$$0 \leq c_1 \log_a n \leq f(n) \leq c_2 \log_a n.$$ 

But for any $a, b$, we have $\log_a n = \frac{\log_b n}{\log_b a}$. So we have $\forall n \geq n_0,$

$$0 \leq c_1 \frac{\log_b n}{\log_b a} \leq f(n) \leq c_2 \frac{\log_b n}{\log_b a}.$$ 

Noting that $\log_b a$ is a constant, we can define

$$c_3 = \frac{c_1}{\log_b a}, \quad c_4 = \frac{c_2}{\log_b a}.$$ 

We now have $\forall n \geq n_0,$

$$0 \leq c_3 \log_b n \leq f(n) \leq c_4 \log_b n.$$ 

Hence by definition, $f(n) \in \Theta(\log_b n)$.

**Alternative Solution:**
Let $f(n)$ be an arbitrary function such that $f(n) \in \Theta(\log_a n)$. Then by the limit rule, we know that

$$\lim_{n \to \infty} \frac{f(n)}{\log_a n} = C$$

for some constant $C$ such that $0 < C < \infty$. We also have

$$\lim_{n \to \infty} \frac{\log_a n}{\log_b n} = \lim_{n \to \infty} \frac{\log_a n}{\log_a n / \log_b a} = \lim_{n \to \infty} \log_a b = \log_a b,$$

and $0 < \log_a b < \infty$. Hence we have

$$\lim_{n \to \infty} \frac{f(n)}{\log_b n} = \lim_{n \to \infty} \frac{f(n)}{\log_a n} \cdot \frac{\log_a n}{\log_b n} = \left( \lim_{n \to \infty} \frac{f(n)}{\log_a n} \right) \cdot \left( \lim_{n \to \infty} \frac{\log_a n}{\log_b n} \right) = C \cdot \log_a b,$$

and $0 < C \cdot \log_a b < \infty$. Hence by the limit rule, $f(n) \in \Theta(\log_b n)$. A similar argument can be used to show that for an arbitrary $f(n) \in \Theta(\log_b n)$, it must be the case that $f(n) \in \Theta(\log_a n)$. □
4. Prove that $f(n) = 2n^2 + 10 \notin O(n)$.

**Solution:** Our proof will be by contradiction. Assume that $f(n) \in O(n)$. Then by definition, there exist constants $c, n_0 > 0$ such that $\forall n \geq n_0, 0 \leq f(n) \leq cn$. Let $n$ be some value such that $n > n_0$ and $n > c/2$. Then

$$f(n) = 2n^2 + 10 \geq 2n(c/2) + 10 = cn + 10 > cn.$$ 

This is a contradiction. Hence our assumption is incorrect and we conclude that $f(n) = 2n^2 + 10 \notin O(n)$.  

**Alternative Solution:** We will use the limit rule. Let $g(n) = n$. Then we have

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{2n^2 + 10}{n} = \lim_{n \to \infty} \left(2n + \frac{10}{n}\right) = \infty.$$ 

This tells us that $f(n) \in \omega(g(n))$; a function cannot be in both $\omega(g(n))$ and $O(g(n))$, so we conclude that $f(n) \notin O(g(n))$.  